PHYS 532 L35

<u>Cherenkov Radiation</u> - A conical wave front emitted by a charged particle moving through a medium with speed $> c/\sqrt{\mu\epsilon}$ = phase speed of EM waves in the medium.
(An electromagnetic shock wave.)

Substitute $\quad \underline{B} = \nabla \times \underline{A}$ (6.7)

and $\quad \underline{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$ (6.8)

into the inhomogeneous Maxwell equations

$$\left.\begin{aligned} \nabla \times \underline{H} &= \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{D}}{\partial t} \\ \nabla \cdot \underline{D} &= 4\pi\rho \end{aligned}\right\} \quad (6.6)$$

with $\underline{H} = \underline{B}/\mu$ and $\underline{D} = \epsilon \underline{E}$ to get

$$\frac{1}{\mu}\nabla \times (\nabla \times \underline{A}) = \frac{4\pi}{c}\underline{J} + \frac{\epsilon}{c}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}\right)$$

$$\epsilon\, \nabla \cdot \left(-\nabla\Phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}\right) = 4\pi\rho \qquad (\epsilon = \text{const})$$

Using the Lorentz gauge condition $\nabla \cdot \underline{A} + \frac{\epsilon\mu}{c}\frac{\partial \Phi}{\partial t} = 0$, this gives the inhomogeneous wave eqns.

$$\left(\nabla^2 - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\underline{A} = -\frac{4\pi\mu}{c}\underline{J} \qquad (\text{x } 13.14)$$

$$\left(\nabla^2 - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{4\pi\rho}{\epsilon} \qquad (\text{x } 13.15)$$

(We've done this before.) Now assume $\mu = 1$ but $\epsilon \neq 1$. Define the Fourier transform

$$\Phi(\underline{k},\omega) = \frac{1}{(2\pi)^2}\int d^3x \int dt\, e^{-i(\underline{k}\cdot\underline{x} - \omega t)}\, \Phi(\underline{x},t) \qquad (x13.16a)$$

such that
$$\Phi(\underline{x},t) = \frac{1}{(2\pi)^2}\int d^3k \int d\omega\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}\, \Phi(\underline{k},\omega) \qquad (x13.16b)$$

And similarly for $\underline{A}$, $\underline{J}$, and ρ. The Fourier-transformed versions of (x13.14) and (x13.15) are

$$\left.\begin{aligned} \left(-k^2 + \frac{\epsilon}{c^2}\omega^2\right)\Phi(\underline{k},\omega) &= -\frac{4\pi}{\epsilon}\rho(\underline{k},\omega) \\ \left(-k^2 + \frac{\epsilon}{c^2}\omega^2\right)\underline{A}(\underline{k},\omega) &= -\frac{4\pi}{c}\underline{J}(\underline{k},\omega) \end{aligned}\right\} \qquad (13.22)$$

In a source-free region (13.22) $\rightarrow$ EM waves with phase speed $\omega/k = c/\sqrt{\epsilon}$ (Generally $< c$).

Assume a point charge q moving uniformly in the x direction:
$$\rho(\underline{x},t) = q\,\delta(y)\delta(z)\delta(x - vt) \qquad (x13.17)$$

$$\left.\begin{aligned} \rightarrow \rho(\underline{k},\omega) &= \frac{q}{(2\pi)^2}\int d^3x \int dt\, e^{-i(\underline{k}\cdot\underline{x} - \omega t)}\,\delta(y)\delta(z)\delta(x - vt) \\ &= \frac{q}{(2\pi)^2}\int dt\, e^{-ik_x vt + i\omega t} = \frac{q}{2\pi}\delta(\omega - k_x v) \\ \text{and}\quad \underline{J}(\underline{k},\omega) &= \rho(\underline{k},\omega)\, v\hat{x} \end{aligned}\right\} (x13.18)$$

Substituting (x13.18) in (13.22) gives

$$\left.\begin{aligned} \Phi(\underline{k},\omega) &= \frac{2q\,\delta(\omega - k_x v)}{\epsilon\left(k^2 - \epsilon\frac{\omega^2}{c^2}\right)} \\ \underline{A}(\underline{k},\omega) &= \epsilon\frac{v}{c}\hat{x}\,\Phi(\underline{k},\omega) \end{aligned}\right\} \qquad (13.25)$$

Substituting (13.25) in (6.8) gives

$$\underline{E}(\underline{k},\omega) = -i\underline{k}\,\Phi(\underline{k},\omega) + \frac{i\omega}{c}\underline{A}(\underline{k},\omega)$$

$$= \left(-i\underline{k} + \frac{i\omega}{c}\frac{\epsilon v}{c}\hat{x}\right)\Phi(\underline{k},\omega) \qquad (13.26a)$$

Now let's calculate the field component E_x (along the direction of motion) at a point $(x,y,z) = (0,b,0)$, i.e. a distance b from the particle's path:

$$E_x(0,b,0,\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{ik_y b} E_x(\underline{k},\omega)$$

$$= \frac{i}{(2\pi)^{3/2}}\int d^3k\, e^{ik_y b}\left(\frac{\epsilon\omega v}{c^2} - k_x\right)\frac{2q\,\delta(\omega - k_x v)}{\epsilon\left(k^2 - \epsilon\frac{\omega^2}{c^2}\right)} \qquad (\times 13.19)$$

The integral over k_x can be done immediately, using the δ function:

$$E_x(0,b,0,\omega) = \frac{2iq}{(2\pi)^{3/2}\epsilon v}\int dk_y\int dk_z\, \frac{e^{ik_y b}\left(\frac{\omega\epsilon v}{c^2} - \frac{\omega}{v}\right)}{\left[\left(\frac{\omega}{v}\right)^2 + k_y^2 + k_z^2 - \epsilon\frac{\omega^2}{c^2}\right]}$$

$$= \frac{-2iq\omega}{(2\pi)^{3/2}\epsilon v^2}\left(1 - \epsilon\beta^2\right)\int dk_y\int dk_z\,\frac{e^{ik_y b}}{k_y^2 + k_z^2 + \lambda^2}$$

where $\beta = v/c$ as usual, and $\qquad (\times 13.20)$

$$\lambda^2 \equiv \frac{\omega^2}{v^2}\left(1 - \epsilon\beta^2\right) \qquad (13.30)$$

The integral over k_z is only slightly harder:

$$\int_{-\infty}^{\infty} \frac{dk_z}{k_y^2+k_z^2+\lambda^2} = \frac{1}{\sqrt{k_y^2+\lambda^2}} \tan^{-1}\left(\frac{k_z}{\sqrt{k_y^2+\lambda^2}}\right)\Bigg|_{k_z=-\infty}^{\infty}$$

$$= \frac{\pi}{\sqrt{k_y^2+\lambda^2}}$$

Finally, the integral over k_y is hard, but fortunately it has a name — it is an integral representation of a modified Bessel function:

$$\int_{-\infty}^{\infty} \frac{e^{ik_y b}}{\sqrt{k_y^2+\lambda^2}}\, dk_y = 2K_0(\lambda b)$$

So we have altogether

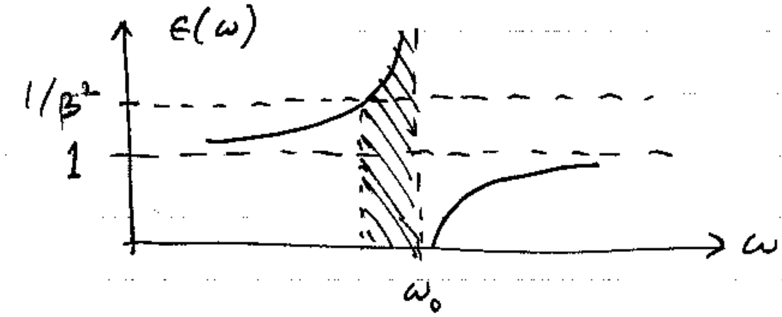
$$E_x(0,b,0,\omega) = -\frac{iq\omega}{\epsilon v^2}\sqrt{\frac{2}{\pi}}\,(1-\epsilon\beta^2)\,K_0(\lambda b) \qquad (13.32)$$

Similar expressions can be obtained in the same way for $E_y(0,b,0,\omega)$ and $B_z(0,b,0,\omega)$ (13.33).

For the usual case $\lambda^2 > 0$ $(v < c/\sqrt{\epsilon})$, these expressions can be used to calculate the energy lost to the medium per unit distance (see Jackson § 13.3). This is complicated but straightforward.

However, if $\lambda^2 < 0$ (possible for a relativistic particle in a dielectric medium), something interesting happens.

$\lambda^2 < 0$ means $v > c/\sqrt{\epsilon}$ = phase speed of light in the medium. This condition requires $\epsilon > 1/\beta^2 \gtrsim 1$. Recall from chapt. 7 (L15) that $\epsilon(\omega)$ near a resonance frequency ω_0 looks like



so, for a sufficiently relativistic particle ($\beta \sim 1$), there is a "Cherenkov band" (shaded), just below each resonance frequency, where $\epsilon > 1/\beta^2$.

$\therefore$ Ch-rad. occurs in distinct frequency bands, related to the atomic resonance frequencies of the medium.

Moreover, it produces a conical wavefront, like a shock wave, with the particle at the cone's apex. To see this, we can use (13.32) with the asymptotic approximation

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad (x \gg 1) \qquad (3.104)$$

to get

$$E_x(0,b,0,\omega) \approx \frac{iq\omega}{c^2}\left(1 - \frac{1}{\epsilon\beta^2}\right)\frac{e^{-\lambda b}}{\sqrt{\lambda b}} \qquad (13.45)$$

The corresponding expression for E_y

$$E_y(0,b,0,\omega) = \frac{q}{v}\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\epsilon}\,K_1(\lambda b) \qquad (13.33)$$

(not derived here) can be combined with the same asymptotic approximation

$$K_1(x) \approx K_0(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad (x \gg 1)$$

to get

$$E_y(0,b,0,\omega) = \frac{q\lambda}{\epsilon v}\,\frac{e^{-\lambda b}}{\sqrt{\lambda b}} \qquad (13.45)$$

Because we are considering the case $\lambda^2 < 0$, the exponentials in (13.45) become sinusoidal oscillations in space with wavelength $= 2\pi/\lambda$. (Sorry about the notation.) These oscillations cancel when we take the ratio

$$\frac{E_x}{E_y} = \frac{\frac{iq\omega}{\epsilon v^2}(\epsilon\beta^2-1)}{\frac{q\lambda}{\epsilon v}}$$

$$= \frac{i\omega}{v\lambda}(\epsilon\beta^2-1) = \frac{i\omega(\epsilon\beta^2-1)}{v\,\frac{\omega}{v}\sqrt{1-\epsilon\beta^2}}$$

$$= -\sqrt{\epsilon\beta^2-1} \qquad (X13.21)$$

Note that both E_x and E_y (13.45) are $\propto 1/\sqrt{b}$. This is characteristic of radiation from a line source.

The electric field of the wave is parallel to the wave front (because it propagates in the $\underline{E} \times \underline{B}$ direction), so the angle θ_c between the wave-front normal and the direction of motion $\hat{x}$ is given by

$$\tan\theta_c = -\frac{E_x}{E_y} = \sqrt{\epsilon\beta^2 - 1} \qquad (13.49)$$

and the trig. identity $\cos^2 x = \frac{1}{1+\tan^2 x}$ gives

$$\cos^2\theta_c = \frac{1}{\epsilon\beta^2} \quad \text{or} \quad \boxed{\cos\theta_c = \frac{c}{\sqrt{\epsilon}\, v}} \qquad (13.50)$$

We can also see this graphically with the help of Jackson's Fig. 13.15:

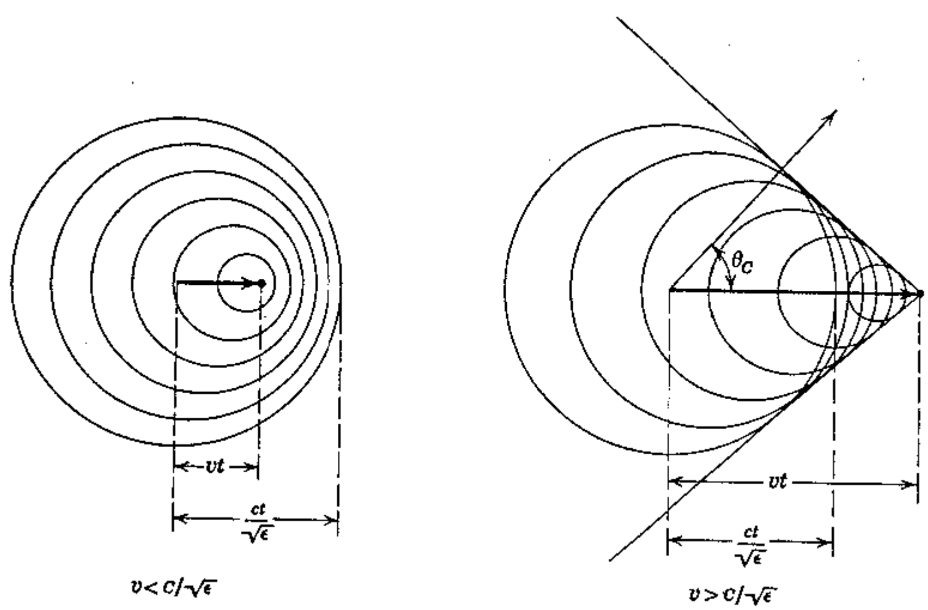


Figure 13.5 Cherenkov radiation. Spherical wavelets of fields of a particle traveling less than and greater than the velocity of light in the medium. For $v > c/\sqrt{\epsilon}$, an electromagnetic "shock" wave appears, moving in the direction given by the Cherenkov angle θ_C.

Like a shock wave.