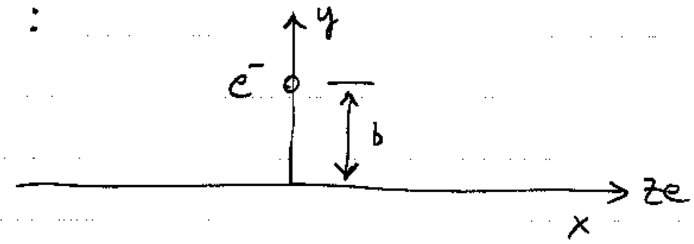
PHYS 532 L34

Collisions, Energy Loss, and Scattering of Charged Particles

Coulomb collisions (This departs from Jackson §13.1, but you should probably read that anyway.)

Consider a charge ze approaching an electron with impact parameter b:



Assume a weak collision, such that ze moves approximately on a straight line. Assume the electron also doesn't move much from its original location $(x, y, z) = (0, b, 0)$. Let $t = 0$ be the time of closest approach. In L31 we derived the electric field in the electron's frame due to the passing charge ze:

$$E_y = \frac{\gamma b (ze)}{\left[b^2 + \gamma^2 v^2 t^2\right]^{3/2}} \qquad (11.152)$$

The momentum transferred to the electron is

$$\Delta p_y = -\int_{-\infty}^{\infty} e E_y \, dt = -\gamma b z e^2 \int_{-\infty}^{\infty} \frac{dt}{\left[b^2 + \gamma^2 v^2 t^2\right]^{3/2}}$$

$$= \frac{-\gamma z e^2}{\gamma v b} \underbrace{\int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{3/2}}}_{=2} = \frac{-2 z e^2}{b v} \qquad (X13.1)$$

$[u = \gamma v t / b]$

By conservation of momentum, the incident particle gains transverse momentum of the same magnitude, so its angular deflection is

$$\Delta\theta \approx \frac{\Delta p_y}{p} = \frac{2ze^2}{bvp} \qquad (x13.2)$$

The differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = \left|\frac{2\pi b\, db}{2\pi \sin\theta\, d\theta}\right| \approx \left|\frac{b\, db}{\theta\, d\theta}\right|$$

From (x13.2), $b = \frac{2ze^2}{vp\,\theta} \rightarrow \frac{db}{d\theta} = \frac{-2ze^2}{vp\,\theta^2}$

$$\rightarrow \frac{d\sigma}{d\Omega} \approx \left|\frac{1}{\theta}\, b\, \frac{db}{d\theta}\right| = \frac{4z^2e^4}{p^2v^2\theta^4} \qquad (x13.3)$$

which is ≈ the Rutherford scattering cross section (13.1) for $\theta \ll 1$.

Note that (x13.2) has a problem in the limit $b \to 0$, and (x13.3) has a problem in the opposite limit $\theta \to 0$. Thus the weak collision approximation imposes limits on both maximum and minimum allowed values of the impact parameter.

Instead of a single electron, assume the charge ze moves through a medium with N electrons per unit volume. Assume its direction θ undergoes a random walk as the result of many weak collisions:

$$\Delta\theta^2 = \int 2\pi b\,db\,\Delta x\,N\left(\frac{2ze^2}{bvp}\right)^2 = \frac{8\pi\Delta x z^2 e^4 N}{p^2 v^2}\int\frac{db}{b}$$

The integral $\int \frac{db}{b} = \ln b$ diverges both at $b \to 0$ and at $b \to \infty$, so we better not go there. Define a range of impact parameters $b_{min} < b < b_{max}$ where the weak-collision theory is valid.
Then

$$\int \frac{db}{b} \approx \ln\left(\frac{b_{max}}{b_{min}}\right) \equiv \ln\Lambda \qquad (\times 13.4)$$

the famous "Coulomb logarithm".
The details are contained in the number Λ.
Fortunately, we only need its logarithm, which we can usually assume is $\sim 10 \pm$ a factor ~ 3. For example, $\ln(20) \sim 3$ and $\ln(10^{13}) \sim 30$.

There are absolute limits on b_{min}, b_{max}. For example,

b_{min} cannot be $\lesssim \frac{\hbar}{\gamma m v}$, the uncertainty in the particle's position.

b_{min} cannot be $\lesssim \frac{2ze^2}{\gamma m v^2}$, the value for which $\Delta\theta \sim 1$ in $(\times 13.2)$.

b_{max} cannot be $\gtrsim$ the value for which the energy transfer to the target electron is $\lesssim$ its binding energy. (otherwise energy transfer is to atom, not electron.)

b_{max} cannot be $\gtrsim$ Debye length in a plasma. (see below.)

In any case, the energy given up to the target electron is

$$\Delta T_e = \frac{\Delta p_y^2}{2m_e} = \left(\frac{2ze^2}{bv}\right)^2 \frac{1}{2m_e}$$

and the rate of energy loss by the charge ze is

$$\Delta T = -\int 2\pi b\, db\, \Delta x\, N \left[\frac{1}{2m_e}\left(\frac{2ze^2}{bv}\right)^2\right.$$

or

$$\boxed{\frac{dT}{dx} = \frac{-4\pi z^2 e^4 N}{m_e v^2} \ln\Lambda} \qquad (13.7)$$

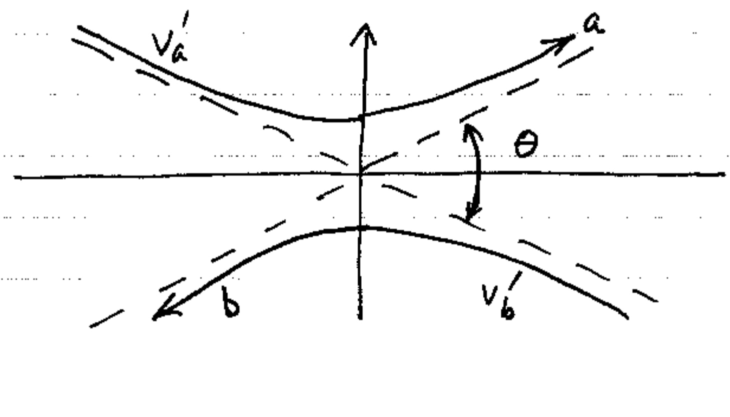
Conductivity in an Unmagnetized Plasma (or $\parallel \underline{B}$)

Consider two particle species a and b drifting at average velocities $\underline{\bar{v}}_a$ and $\underline{\bar{v}}_b$. The average force on species a due to collisions with species b is

$$\underline{\bar{F}}_{a(b)} = -m_a \nu_{ab} (\underline{\bar{v}}_a - \underline{\bar{v}}_b) \qquad (\times 13.5)$$

and vice versa (interchange a, b). This is the definition of the collision frequency ν_{ab}. Let's try to calculate it. (Assume $v_a, v_b \ll c$.)

Consider a typical collision in the center-of-mass (cm) system, which moves at $\underline{V}_{cm} = \dfrac{m_a \underline{v}_a + m_b \underline{v}_b}{m_a + m_b}$ $(\times 13.6)$



The particle velocities in the CM frame are

$$v_a' = \frac{m_b(\underline{v}_a - \underline{v}_b)}{m_a + m_b} \qquad v_b' = \frac{m_a(\underline{v}_b - \underline{v}_a)}{m_a + m_b}$$

(Note $\underline{v}_a' - \underline{v}_b' = \underline{v}_a - \underline{v}_b$.)

θ = scattering angle = change in angle of $\underline{v}_a - \underline{v}_b$.

$$\Delta \underline{v}_a = \frac{-m_b}{m_a + m_b} (1 - \cos\theta)(\underline{v}_a - \underline{v}_b) \qquad (X13.7)$$

(in either lab frame or CM frame)

To evaluate (X13.5) we have to integrate over the velocity distributions of both species. If both are assumed to be Maxwellians at the same temperature T, this gives (skipping the algebra)

$$\frac{d}{dt}(\overline{p_a}) \equiv -m_a \nu_{ab} (\overline{\underline{v}_a} - \overline{\underline{v}_b})$$

$$= -K m_r (\overline{\underline{v}_a} - \overline{\underline{v}_b}) n_b \left(\frac{kT}{m_r}\right)^{1/2} \frac{2}{\pi} \int_0^\infty x^2 \sigma(xkT) e^{-x} dx \qquad (X13.8)$$

where $$\sigma(E) = \int_0^\pi \frac{d\sigma(E)}{d\Omega} (1 - \cos\theta) 2\pi \sin\theta \, d\theta \qquad (X13.9)$$

m_r = reduced mass = $\dfrac{m_a m_b}{m_a + m_b}$

n_b = density of species b $\qquad (K \sim 1)$

Use the Rutherford scattering cross section

$$\frac{d\sigma(E)}{d\Omega} = \left(\frac{Z_a Z_b e^2}{4E}\right)^2 \frac{1}{\sin^4(\frac{\theta}{2})}$$

($\approx$ (13.1) with $pv \to 2E$). Then (X13.9) gives

$$\sigma(E) = \left(\frac{z_a z_b e^2}{4E}\right)^2 4 \int_{\theta_{min}}^{\pi} \frac{\sin\theta(1-\cos\theta)\,d\theta}{\sin^4(\frac{\theta}{2})}$$

Here the lower limit of the integral has been changed from 0 to θ_{min} to avoid a logarithmic divergence. Using trig identities we find the integral is

$$\int_{\theta_{min}}^{\pi} \frac{\sin\theta(1-\cos\theta)\,d\theta}{\sin^4(\frac{\theta}{2})} = \ln\left[\sin\frac{\theta}{2}\right]_{\theta_{min}}^{\pi}$$

$$= \ln\left(\frac{1}{\sin\frac{\theta_m}{2}}\right) \equiv \ln\Lambda$$

and (x13.8) gives

$$\nu_{ab} = \frac{m_r}{m_a}\left(\frac{kT}{m_r}\right)^{1/2} n_b K \int_0^\infty x^2 \left[\frac{z_a^2 z_b^2 e^4}{x^2 (kT)^2}\right] \ln\Lambda\, e^{-x}\,dx$$

$$= K n_b \frac{m_r}{m_a}\left(\frac{kT}{m_r}\right)^{1/2} \frac{z_a^2 z_b^2 e^4}{(kT)^2} \ln\Lambda$$

For electrons scattering off ions, set $m_r \approx m_a = m_e$, $n_b = n_i$, $z_a = -1$, $z_b = z$ $\rightarrow$

$$\boxed{\nu_{ei} = \frac{K n_i z^2 e^4 \ln\Lambda}{m_e^{1/2} (kT)^{3/2}}} \qquad (x13.10)$$

Note that $n_i m_i \nu_{ie} = n_e m_e \nu_{ei}$. (x13.11)

We did not need to assign a value of θ_{max} above, because we used the Rutherford scattering formula that is valid for strong collisions.

Given ν_{ei} we can calculate the conductivity.

Assume a balance between electrostatic force and collisional drag force, for both electrons and ions:

$$-n_e e \underline{E} + n_e m_e (\underline{v}_i - \underline{v}_e)\nu_{ei} = 0 \qquad (X13.12)$$

$$n_i Z e \underline{E} + n_i m_i (\underline{v}_e - \underline{v}_i)\nu_{ie} = 0 \qquad (X13.13)$$

(Note that this, plus the condition (X13.11), gives the quasineutrality condition $n_i Z = n_e$.)

$$(X13.12) \rightarrow \quad \underline{v}_e - \underline{v}_i = \frac{-e\underline{E}}{m_e \nu_{ei}}$$

The current density is $\underline{J} = -n_e e(\underline{v}_e - \underline{v}_i) = \frac{n_e e^2}{m_e \nu_{ei}} \underline{E}$

But the conductivity is defined by $\underline{J} = \sigma \underline{E}$ so

$$\sigma = \frac{n_e e^2}{m_e \nu_{ei}} = \text{"Spitzer conductivity"}$$

$$= \frac{(kT)^{3/2}}{K Z^2 e^2 (m_e)^{1/2} \ln \Lambda}$$

What is θ_{min}?

Minimum deflection corresponds to maximum impact parameter. If the $1/r$ potential extends to ∞, the Coulomb integral diverges.
In a plasma, b_{max} is the <u>Debye length</u>.

A positive charge attracts a cloud of negative charge that shields its potential at large distances.

The actual potential around a charge q in an unmagnetized plasma is not just q/r but

$$\Phi = \frac{q}{r} e^{-r/\lambda_D}$$

where λ_D = Debye length = $\left(\frac{kT}{4\pi n e^2}\right)^{1/2}$

(see PHYS 480)

So the maximum impact parameter is $b_{max} \sim \lambda_D$.

Then (x13.2) gives $\Delta\theta_{min} \sim \frac{2Ze^2}{mv^2 \lambda_D}$