

Dynamics of Relativistic Particles in Electromagnetic Fields (Jackson §12.1-3)

In classical mechanics, the principle of least action $\Rightarrow \int L dt = \text{extremum}$ for the correct path, where $L = T - V = \text{kinetic energy} - \text{potential energy}$. This is for non-relativistic (NR) velocities.

To make the action integral manifestly covariant, we replace dt by $\gamma d\tau$ ($d\tau = \text{proper time element} = \text{Lorentz invariant scalar}$), and make γL a Lorentz-invariant scalar that reduces to the classical L when $\gamma \rightarrow 1$.

$$\text{Try the form } \gamma L = -mc^2 - \frac{q}{c} U_\alpha A^\alpha \quad (\text{x12.1})$$

where $U^\alpha = (\gamma c, \gamma \underline{v}) = p^\alpha / m = 4\text{-velocity}$, and

$$A^\alpha = (\Phi, \underline{A}) = 4\text{-potential}.$$

(x12.1) is clearly a Lorentz invariant scalar.

The question is, does it produce the correct equation of motion

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} U_\nu \quad (\text{x11.70})$$

when used in the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad ? \quad (12.5)$$

Substituting the definitions of γ , u_α , A^α in (12.1) gives

$$L = -mc^2 \sqrt{1 - \dot{\mathbf{q}}^2/c^2} - q\Phi + \frac{q}{c} \dot{\mathbf{q}} \cdot \underline{A} \quad (12.12)$$

where $\dot{\mathbf{q}}^2 \equiv \dot{q}_i \dot{q}_i$, $\Phi = \Phi(q_i)$, $\underline{A} = \underline{A}(q_i)$.

(q_i = cartesian coordinates of particle.)

In the non-relativistic limit, $\sqrt{1 - \dot{\mathbf{q}}^2/c^2} \approx 1 - \frac{1}{2} \dot{\mathbf{q}}^2/c^2$ and

$$L \rightarrow -mc^2 + \frac{m}{2} \dot{\mathbf{q}}^2 - q\Phi + \frac{q}{c} \dot{\mathbf{q}} \cdot \underline{A} \quad (v \ll c)$$

which looks familiar (apart from the constant $-mc^2$, which has no effect). Try (12.12) in (12.5):

$$\frac{d}{dt} \left(\frac{m \dot{q}_i}{\sqrt{1 - \dot{\mathbf{q}}^2/c^2}} \right) + \frac{q}{c} \frac{dA_i}{dt} + q \frac{d\Phi}{dq_i} - \frac{q}{c} \dot{q}_j \frac{dA_j}{dq_i} = 0$$

$$\frac{d}{dt} (\gamma m \dot{q}_i) = -\frac{q}{c} \left(\frac{\partial}{\partial t} + \dot{q}_j \frac{\partial}{\partial q_j} \right) A_i - q \frac{\partial \Phi}{\partial q_i} + \frac{q}{c} \dot{q}_j \frac{\partial A_j}{\partial q_i}$$

Using $\underline{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \underline{A}}{\partial t}$, this can be written

$$\frac{d}{dt} (\gamma m \dot{q}_i) = qE_i + \frac{q}{c} \dot{q}_j \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right) \quad (12.2)$$

$$\text{But } \left[\dot{\mathbf{q}} \times (\nabla \times \underline{A}) \right]_i = \epsilon_{ijk} \dot{q}_j \epsilon_{klm} \frac{\partial A_m}{\partial q_l}$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \dot{q}_j \frac{\partial A_m}{\partial q_l}$$

$$= \dot{q}_j \frac{\partial A_j}{\partial q_i} - \dot{q}_j \frac{\partial A_i}{\partial q_j} \quad (12.3)$$

Combining (x12.2) and (x12.3) gives

$$\frac{d}{dt}(\gamma m \underline{\dot{q}}) = q \left(\underline{E} + \frac{\dot{q}}{c} \times \underline{B} \right) \quad (\text{x12.4})$$

in agreement with (12.1). So the trial form (x12.1) works.

Given the Lagrangian L , we can find the Hamiltonian H in the usual way: $H = p_i \dot{q}_i - L$ where $p_i =$ momentum conjugate to q_i

$$= \frac{\partial L}{\partial \dot{q}_i} = \frac{m \dot{q}_i}{\sqrt{1 - \dot{q}^2/c^2}} + \frac{q}{c} A_i \quad (\text{x12.5})$$

$$\begin{aligned} \Rightarrow H &= \left[\frac{m \dot{q}_i}{\sqrt{1 - \dot{q}^2/c^2}} + \frac{q}{c} A_i \right] \dot{q}_i - \left[-\sqrt{1 - v^2/c^2} mc^2 - q\Phi + \frac{q}{c} \underline{v} \cdot \underline{A} \right] \\ &= \frac{m[v^2 + c^2(1 - v^2/c^2)]}{\sqrt{1 - v^2/c^2}} + q\Phi \end{aligned}$$

$$\boxed{H = \gamma mc^2 + q\Phi} = \text{total energy} \quad (\text{x12.6})$$

($\gamma mc^2 =$ rest energy + kinetic energy, $q\Phi =$ potential energy.)

Conventionally, H is expressed as a function of (q_i, p_i) , not of (q_i, \dot{q}_i) . It can be shown (in your next homework) that (x12.6) can be written

$$H = \sqrt{m^2 c^4 + |c\underline{p} - q\underline{A}|^2} + q\Phi \quad (12.17)$$

In the NR limit, (12.17) becomes

$$H \approx mc^2 + \frac{1}{2m} \left| \underline{P} - \frac{q}{c} \underline{A} \right|^2 + q\Phi$$

Same as classical result, except for mc^2 term, which has no effect in eqns. of motion.

It can also be shown (also in your next homework) that using (12.17) in Hamilton's eqn. of motion gives (12.1) again.

The classical (but relativistic) equation of motion doesn't need Φ or \underline{A} , only their derivatives (\underline{E} and \underline{B}), just as in the NR case.

However, in quantum mechanics, Φ and \underline{A} acquire a physical significance apart from \underline{E} and \underline{B} . For example, they both appear explicitly in H (12.17), which is an operator in Schrödinger's eqn.

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

For example, Aharonov and Bohm (Phys. Rev., 115, 485, 1959) predicted, and Chambers (Phys. Rev. Lett., 5, 3, 1960) confirmed in the lab, that the value of \underline{A} can affect the interference pattern of a split electron beam even if $\underline{B} = 0$ in the path of both beams.

Motion of a charged particle in uniform \underline{B}

Assume $\underline{E} = 0$, $\underline{B} = B_0 \hat{z}$

$$\text{From (x11.71), } \frac{d}{dt}(\gamma m \underline{v}) = \frac{q}{c} \underline{v} \times (B_0 \hat{z}) \quad (\text{x12.7})$$

$$\text{and from (x11.73), } \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \text{constant} \quad (\text{x12.8})$$

$$\Rightarrow \frac{d\underline{v}}{dt} = \omega_B \underline{v} \times \hat{z} \quad (12.38)$$

$$\text{where } \omega_B = \frac{qB_0}{\gamma mc} = \text{"Larmor frequency"} \quad (12.39)$$

(or gyrofrequency or cyclotron frequency)

(12.38) describes circular motion in the xy plane:

$$v_x = v_{\perp} \cos(\omega_B t + \phi)$$

$$v_y = -v_{\perp} \sin(\omega_B t + \phi)$$

where v_{\perp} and ϕ are constants. The radius of the circle is

$$a_c = \frac{v_{\perp}}{\omega_B} = \frac{\gamma m c v_{\perp}}{q B_0} = \text{"Larmor radius"}$$

(or gyroradius or cyclotron radius)

If $v_z (= \text{constant}) \neq 0$, the motion is a helix.

What if $\underline{E} \neq 0$? The effect of \underline{E} depends very much on whether $\underline{E} \parallel \underline{B}$ or $\underline{E} \perp \underline{B}$. (For other angles between \underline{E} and \underline{B} , the effect is a superposition of these two special cases.)

Effect of $\underline{E} \parallel \underline{B}$ ($\parallel \hat{z}$) (assume $\underline{v}_\perp = 0$.)

From (x11.71), $\frac{d}{dt}(\gamma m v_z) = q E_z$ (x12.9)

Assume $v_z = 0$ at $t = 0$. Integrating (x12.9) in time gives

$$\gamma m v_z = \frac{m v_z}{\sqrt{1 - v_z^2/c^2}} = q E_z t$$

$$m^2 v_z^2 = (q E_z t)^2 (1 - v_z^2/c^2)$$

$$v_z^2 \left(m^2 + \frac{q^2 E_z^2 t^2}{c^2} \right) = q^2 E_z^2 t^2$$

$$v_z = \frac{q E_z t}{m \sqrt{1 + \frac{q^2 E_z^2 t^2}{m^2 c^2}}} \quad (\text{x12.10})$$

Particle initially accelerates at a uniform rate $\frac{qE}{m}$, but for large t , $v_z \rightarrow c$ as $t \rightarrow \infty$.
(The usual relativistic speed limit.)

Effect of $\underline{E} \perp \underline{B}$

The simplest way to see the effect of $\underline{E} \perp \underline{B}$ is to transform with a velocity \underline{v} into a frame in which $\underline{E} = 0$. From (11.149)

$$\underline{E}' = \gamma \left(\underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right) = 0$$

Crossing with \underline{B} (on the right) gives

$$\underline{v} = c \frac{\underline{E} \times \underline{B}}{B^2} + \frac{\underline{B}(\underline{v} \cdot \underline{B})}{B^2}$$

The second term is the component of the transformation velocity $\parallel \underline{B}$. It has no effect on \underline{E}' , and we are free to set it = 0 (choose a transformation velocity $\perp \underline{B}$). Then

$$\underline{v} = c \frac{\underline{E} \times \underline{B}}{B^2} \equiv \underline{v}_D = \text{"E} \times \underline{B} \text{ drift"} \quad (12.11)$$

In the primed frame, we have $\underline{E}' = 0$ and, from (11.149),

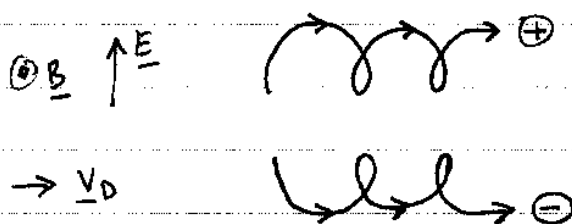
$$\begin{aligned} \underline{B}' &= \gamma \left(\underline{B} - \frac{\underline{v}}{c} \times \underline{E} \right) \\ &= \gamma \underline{B} \left(1 - \frac{E^2}{B^2} \right) = \gamma \underline{B} \left(1 - \frac{v^2}{c^2} \right) = \frac{\underline{B}}{\gamma} \end{aligned}$$

($\underline{B}' \approx \underline{B}$ to first order in v/c .)

In the primed frame, the motion is simple gyration about \underline{B}' (as derived above) with frequency (from 12.39)

$$\omega_B' = \frac{qB'}{\gamma mc} = \frac{qB}{\gamma^2 mc} \approx \omega_B \quad (v \ll c)$$

In the original frame, the motion is a superposition of the constant $\underline{E} \times \underline{B}$ drift and the circular gyration. It is a cycloid:



During alternate phases of the cycle, the particle is accelerated / decelerated by \underline{E} so the Larmor radius is larger at the top (bottom) for a positive (negative) charge. In both cases the result is a net drift to the right at $c \underline{E} \times \underline{B} / B^2$. At this drift speed, the average magnetic force $q \frac{\underline{v}}{c} \times \underline{B}$ exactly balances the electric force $q \underline{E}$.

This assumes that $\frac{v_D}{c} = \frac{E}{B} < 1$, i.e., $E < B$.

This is usually true in nature, but not necessarily.

If $E > B$, we cannot transform the electric field to zero. In that case, we can instead transform the magnetic field to zero, using a transformation velocity $\underline{v} = c \frac{\underline{E} \times \underline{B}}{E^2}$ with $\frac{v}{c} = \frac{B}{E} < 1$.

In the moving frame, the particle accelerates linearly in the direction of $q \underline{E}$, as above, but in a reduced electric field $\underline{E}' = \underline{E} / \gamma$.

If an external force \underline{F} is applied $\perp \underline{B}$, a similar drift velocity results:

$$\begin{array}{ccc} \odot \underline{B} \uparrow \underline{F} & & \text{Diagram of drift} \\ \rightarrow \underline{v}_D = c \frac{\underline{F} \times \underline{B}}{q B^2} & & \begin{array}{l} \text{Positive charge } \oplus \text{ drifts right} \\ \text{Negative charge } \ominus \text{ drifts left} \end{array} \end{array}$$

for example, magnetic gradient and curvature drifts, which are important in plasma physics [see PHYS 480].