Conservation of Momentum and Energy for free particle

In Newtonian mechanics, a particle subject to no external force obeys

$$\underline{p} = m\frac{d\underline{x}}{dt} = \text{constant} \qquad (XII.61)$$

This looks like it might be the 3 spatial components of a 4-vector, but it isn't. We worked out the transformation for $d\underline{x}/dt$ and it wasn't simple. Suppose we replace $d\underline{x}/dt$ by $d\underline{x}/d\tau$ where $d\tau$ = scalar. Then

$$\left(\frac{d(ct)}{d\tau}, \frac{d\underline{x}}{d\tau}\right) = 4\text{-vector}$$

Is there a scalar $d\tau$ that reduces to dt for $v \ll c$? Yes, the <u>proper time</u> interval $d\tau$:

$$d\tau^2 \equiv \frac{ds^2}{c^2} = dt^2 - \frac{dx^2+dy^2+dz^2}{c^2} = dt^2\left(1-\frac{v^2}{c^2}\right)$$
(XII.62)

$c\,d\tau = (c\,dt, dx, dy, dz)$ = displacement between 2 events in the particle's frame

$d\tau$ = proper time interval between events = time interval measured in the frame where the events are colocated.

$\therefore$ Define $$p^\mu = m\frac{dx^\mu}{d\tau} = mc\frac{dx^\mu}{ds} \qquad (XII.63)$$

If the 3 spatial components of p^μ are conserved, then presumably p^0 is also conserved:

$$p^0 = mc\frac{dx^0}{ds} = mc^2\frac{dt}{ds} = \frac{mc}{\sqrt{1-v^2/c^2}}$$

For $v \ll c$, $p^0 \simeq mc\left(1+\frac{v^2}{2c^2}+\dots\right)$

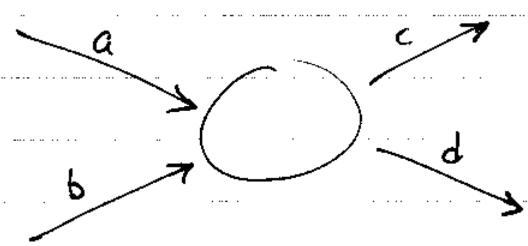
$$p^0 c \simeq mc^2 + \tfrac{1}{2}mv^2 = mc^2 + W_K \qquad (\text{x}11.64)$$

(W_K = kinetic energy of particle)

Einstein interpreted p^0c as the particle energy.
$\therefore$ p^μ conserved $\Rightarrow$ both momentum and energy conserved.

For a free particle, kinetic energy W_K is conserved, and so, therefore, is W_K + (any constant).
What's so special about mc^2, compared to any other arbitrary constant?

Consider an inelastic collision of 2 particles:



Newton says $\underline{p}_a + \underline{p}_b = \underline{p}_c + \underline{p}_d$

<u>and</u> $W_{Ka} + W_{Kb} = W_{Kc} + W_{Kd} + \Delta V$

where ΔV = change of internal energy of particles.

("Inelastic" means $\Delta V \neq 0$.)

Einstein says the collision should satisfy the manifestly covariant expression

$$p_a^\mu + p_b^\mu = p_c^\mu + p_d^\mu \qquad (x11.65)$$

The $\mu = 0$ component of (x11.65) is

$$\gamma_a m_a c^2 + \gamma_b m_b c^2 = \gamma_c m_c c^2 + \gamma_d m_d c^2 \qquad (x11.66)$$

or

$$W_{Ka} + W_{Kb} \approx W_{Kc} + W_{Kd} + \left[(m_c + m_d) - (m_a + m_b)\right] c^2 \qquad (x11.67)$$

so (x11.65) implies that particle mass is not conserved in an inelastic collision. (!) This was a revolutionary hypothesis, a prediction that was subsequently confirmed (over and over).

<u>Motion of a Charged Particle in EM field</u>

The non-relativistic equation of motion is

$$\frac{d\underline{p}}{dt} = q\left(\underline{E} + \frac{\underline{v}}{c} \times \underline{B}\right) \qquad (x11.68)$$

By analogy with the free-particle case, we guess that, to write this in manifestly covariant form, we probably want to replace $\frac{d\underline{p}}{dt}$ by $\frac{dp^\mu}{d\tau}$.

We also need to replace the right side of (x11.68) by a contravariant vector that is a product of $F^{\mu\nu}$ and a manifestly covariant form that resembles velocity. Try

$$q\,\frac{F^{\mu\nu}}{c}\,\frac{dx_\nu}{d\tau} = \frac{q}{c}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} d(ct)/d\tau \\ -dx/d\tau \\ -dy/d\tau \\ -dz/d\tau \end{pmatrix} \qquad (x11.69)$$

The $\mu = 1$ component is

$$q\,\frac{F^{1\nu}}{c}\,\frac{dx_\nu}{d\tau} = \frac{q}{c}\left(cE_x\frac{dt}{d\tau} + B_z\frac{dy}{d\tau} - B_y\frac{dz}{d\tau}\right)$$

$$= q\left(\underline{E}\,\frac{dt}{d\tau} + \frac{1}{c}\frac{d\underline{x}}{d\tau}\times\underline{B}\right)_x$$

$$= \frac{dt}{d\tau}\times\left(\text{right side of } x11.68\right)_x$$

Similarly for the 2 and 3 components. This suggests the manifestly covariant form

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}\,F^{\mu\nu}\,\frac{dx_\nu}{d\tau} \qquad (x11.70)$$

The 3 spatial components of (x11.70) are

$$\frac{dp^i}{d\tau} = \frac{q}{c}\,F^{i\nu}\,\frac{dx_\nu}{d\tau} = \frac{q}{c}\,\frac{dt}{d\tau}\left(c\underline{E} + \underline{v}\times\underline{B}\right)_i$$

or, multiplying by $d\tau/dt$,

$$\frac{dp^i}{dt} = q\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right)_i$$

This looks almost exactly like the non-relativistic equation (x 11.68), except

$$p^i = m\frac{dx^i}{d\tau} = \gamma m\left(\frac{d\underline{x}}{dt}\right)_i = \gamma m\underline{v}_i$$

Thus the assumed manifestly covariant form (x11.70) leads to the momentum equation

$$\frac{d\underline{p}}{dt} = q\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right) \qquad (x11.71)$$

same as (x11.68), but with

$$\underline{p} = \gamma m\underline{v} = \text{relativistic momentum} \qquad (x11.72)$$

(x11.71) is relativistically correct.

The $\mu = 0$ component of (x11.70) is

$$\frac{dp^0}{d\tau} = \frac{q}{c}F^{0\nu}\frac{dx_\nu}{d\tau} = \frac{q}{c}\left(E_x\frac{dx}{d\tau} + E_y\frac{dy}{d\tau} + E_z\frac{dz}{d\tau}\right)$$

$$\frac{d(p^0c)}{d\tau} = q(\underline{E}\cdot\underline{v})\frac{dt}{d\tau}$$

$$\text{or}\quad \frac{d}{dt}\left(\gamma mc^2\right) = q\,\underline{E}\cdot\underline{v} \qquad (x11.73)$$

same as non-relativistic energy equation, but with W_K replaced by γmc^2 = relativistic energy.