We showed last time that the antisymmetric contravariant 2^{nd} rank tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (11.137)$$

satisfies

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad (11.141)$$

which contains the two inhomogeneous Maxwell equns., where

$$J^\nu \equiv (c\rho, \underline{J}) \qquad (11.128)$$

To incorporate the other two (homogeneous) M.E.'s, it is convenient to have the covariant form of $F^{\mu\nu}$. Lower the indices one at a time:

$$F^\mu{}_\nu = F^{\mu\lambda} g_{\lambda\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F_{\eta\nu} = g_{\eta\mu} F^{\mu}{}_{\nu} = g_{\eta\mu} F^{\mu\lambda} g_{\lambda\nu} \qquad (11.138)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F_{\eta\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (11.138)$$

Note that $F_{\mu\nu}$ is obtained from $F^{\mu\nu}$ by replacing $\underline{E}$ by $-\underline{E}$. It is likewise antisymmetric.

We can use $F_{\mu\nu}$ to write the two homogeneous M.E.'s

$$\nabla \cdot \underline{B} = 0 \qquad (x11.3)$$

$$\nabla \times \underline{E} + \frac{1}{c}\frac{\partial \underline{B}}{\partial t} = 0 \qquad (x11.4)$$

$$(x11.3) \Rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\text{or} \quad -\frac{\partial F_{23}}{\partial x^1} - \frac{\partial F_{31}}{\partial x^2} - \frac{\partial F_{12}}{\partial x^3} = 0$$

$$(x11.4)_x \Rightarrow \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t} = 0$$

$$\text{or} \quad -\frac{\partial F_{30}}{\partial x^2} - \frac{\partial F_{02}}{\partial x^3} - \frac{\partial F_{23}}{\partial x^0} = 0$$

Similar results (with different indices) come from the y and z components of (XII.4). Thus (XII.3) and (XII.4) can be summarized by

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \qquad (11.143)$$

where (α, β, γ) = any three of $(0, 1, 2, 3)$.

<u>Conclusion</u>: The four Maxwell equations can be written in manifestly covariant form as

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \qquad &(11.141) \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0 \qquad &(11.143) \end{aligned}}$$

What about the scalar and vector potentials, Φ and $\underline{A}$? If we define the 4-vector

$$\boxed{A^\mu = (\Phi, \underline{A})} \qquad (11.132)$$

then the relationship between $(\underline{E}, \underline{B})$ and $(\Phi, \underline{A})$ can be summarized by the manifestly covariant eqn.

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} \qquad (11.136)$$

Let's check this for a few values of μ, ν:

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1}$$

$$-E_x = \frac{\partial A_x}{c\partial t} + \frac{\partial \Phi}{\partial x}$$

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \qquad \left(\underline{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}\right)$$

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2}$$

$$-B_z = \frac{-\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad (\underline{B} = \nabla \times \underline{A}) \quad \checkmark$$

Also, the Lorentz gauge condition $\boxed{\nabla \cdot \underline{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0}$ can be written

$$\boxed{\frac{\partial A^\mu}{\partial x^\mu} = 0} \qquad (XII.57)$$

which is manifestly covariant. Thus the Lorentz gauge condition is a Lorentz invariant (hence the name).

(The Coulomb gauge condition $\nabla \cdot \underline{A} = 0$ is not.)

Lorentz Transformations of $\underline{E}$ and $\underline{B}$

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} F^{\alpha\beta} \frac{\partial x'^\nu}{\partial x^\beta} \qquad (XII.58)$$

$$\begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y' \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma\beta E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\gamma\beta E_x & -B_z & B_y \\ \gamma E_y - \gamma\beta B_z & -\gamma\beta E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z + \gamma\beta B_y & -\gamma\beta E_z - \gamma B_y & B_x & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -\gamma(E_y-\beta B_z) & -\gamma(E_z+\beta B_y) \\ E_x & 0 & -\gamma(B_z-\beta E_y) & \gamma(B_y+\beta E_z) \\ \gamma(E_y-\beta B_z) & \gamma(B_z-\beta E_y) & 0 & -B_x \\ \gamma(E_z+\beta B_y) & \gamma(-B_y-\beta E_z) & B_x & 0 \end{pmatrix}$$

$$E_x' = E_x \rightarrow \qquad E_\parallel' = E_\parallel$$

$$\left.\begin{aligned} E_y' &= \gamma(E_y - \beta B_z) \\ E_z' &= \gamma(E_z + \beta B_y) \end{aligned}\right\} \rightarrow \qquad \underline{E}_\perp' = \gamma(\underline{E}_\perp + \underline{\beta}\times\underline{B}) \qquad (11.148)$$

$$B_x' = B_x \rightarrow \qquad B_\parallel' = B_\parallel$$

$$\left.\begin{aligned} B_y' &= \gamma(B_y + \beta E_z) \\ B_z' &= \gamma(B_z - \beta E_y) \end{aligned}\right\} \rightarrow \qquad \underline{B}_\perp' = \gamma(\underline{B}_\perp - \underline{\beta}\times\underline{E})$$

The corresponding Galilean transformations are

$$\underline{E}' = \underline{E} + \underline{\beta}\times\underline{B} \qquad \underline{B}' = \underline{B} \qquad (\times 11.59)$$

The parallel components transform the same under LT and GT. The relativistic corrections are in the $\perp$ components, including the ubiquitous γ factor and the substraction of $\underline{\beta}\times\underline{E}$ from $\underline{B}$.

Typically, $E \sim \beta B$, so $\frac{\beta E}{B} \sim \beta^2$.

Thus the change in $\underline{B}$ from one frame to another is 2^{nd} order in $\beta = V/c$.
(The change in $\underline{E}$ is 1^{st} order.)

Coulomb Field of a Point Charge Moving with constant $\underline{V} = v\hat{x}$

Let primed frame = particle rest frame

$$\rightarrow \quad \underline{E}' = \frac{q\,(x'\hat{x} + y'\hat{y} + z'\hat{z})}{(x'^2 + y'^2 + z'^2)^{3/2}} \qquad (x11.60)$$

$$\underline{B}' = 0$$

Transform to lab frame: $x' = \gamma(x - Vt)$, $y' = y$, $z' = z$

$$\underline{E}' = \frac{q\,[\gamma(x-Vt)\hat{x} + y\hat{y} + z\hat{z}]}{[\gamma^2(x-Vt)^2 + y^2 + z^2]^{3/2}}$$

Using (11.148),

$$E_x = E_x' = \frac{q\gamma(x - vt)}{[\gamma^2(x-Vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_y = \gamma\left[E_y' - (\underline{\beta} \times \underline{B}')_y\right] = \gamma E_y' = \frac{q\gamma y}{[\gamma^2(x-Vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_z = \gamma\left[E_z' - (\underline{\beta} \times \underline{B}')_z\right] = \gamma E_z' = \frac{q\gamma z}{[\gamma^2(x-Vt)^2 + y^2 + z^2]^{3/2}}$$

$\underline{E}$ is radially away from charge in all directions, but not isotropic (magnitude depends on direction).

Let $\quad x - Vt = r\cos\theta \quad , \quad \sqrt{y^2 + z^2} = r\sin\theta$

(θ = angle between $\underline{V}$ and observation direction.)

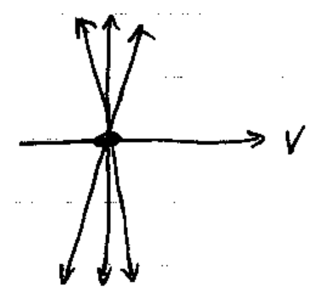
$$E^2 = E_x^2 + E_y^2 + E_z^2 = \frac{q^2\gamma^2}{(\gamma^2\cos^2\theta + \sin^2\theta)^3 r^4}$$

$$= \frac{q^2\gamma^2}{r^4\left[1+(\gamma^2-1)\cos^2\theta\right]^3}$$

$$E\left(\tfrac{\pi}{2}\right) = \frac{q\gamma}{r^2} \qquad E(0) = \frac{q}{\gamma^2 r^2}$$

$$E(\pi/2) \,/\, E(0) = \gamma^3$$

For an ultrarelativistic particle ($\gamma \gg 1$), the Coulomb field is concentrated in a sharp pulse $\perp$ to its direction of motion.



We will see later that the radiation field of an ultrarelativistic particle that is accelerating is concentrated in a headlight beam in the direction of motion – very different from above.

Both are relativistic effects.