PHYS 532 L30

Final notes on manipulating Cartesian tensors in 4-space:

Recall metric tensor $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

(covariant form)

Contravariant form:

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = g_{\mu\nu} \quad (g \text{ is unitary.})$$

$\therefore$ Mixed form: $g_{\mu}^{\nu} = g_{\mu\alpha} g^{\alpha\nu} = \delta_{\mu}^{\nu}$ = Kronecker delta

$$= \begin{Bmatrix} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{Bmatrix}$$

We will also need the "completely antisymmetric 4th-rank tensor"

$$\epsilon_{\mu\nu\eta\sigma} \equiv \begin{cases} 1 & \text{if } \mu\nu\eta\sigma = \text{even permutation of } 0,1,2,3 \\ -1 & \text{if } \mu\nu\eta\sigma = \text{odd} \quad '' \quad '' \quad '' \\ 0 & \text{otherwise} \end{cases}$$

If a contravariant vector is defined by $A^{\mu} = (A^0, \underline{A})$ where $\underline{A} = (A^1, A^2, A^3)$, then its covariant counterpart is $A_{\nu} = g_{\mu\nu} A^{\mu} = (A^0, -\underline{A})$.

Thus the scalar product of two vectors A, B is

$$B \cdot A \equiv B_{\alpha} A^{\alpha} = B^0 A^0 - \underline{B} \cdot \underline{A}$$

Differentiation of a scalar by a contravariant vector gives a covariant vector and vice-versa. We can write this

$$\left.\begin{aligned}\partial_\alpha &\equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right) \\ \partial^\alpha &\equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x_0}, -\nabla\right)\end{aligned}\right\} \quad (11.76)$$

The divergence of a 4-vector is a scalar (invariant):

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \underline{A} \qquad (11.77)$$

The 4-dimensional Laplacian operator is

$$\square \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0 2}} - \nabla^2 \qquad (11.78)$$

$$= \text{wave equation operator.}$$

Now (finally!), let's write the laws of electrodynamics in manifestly covariant form (i.e., in terms of scalars, 4-vectors, and 4-tensors).

Start with the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0 \qquad (5.2)$$

A Galilean transformation gives $\rho' = \rho$, $\underline{J}' = \underline{J} + \rho \underline{V}$. This must be $\approx$ right for $V \ll c$, but a Lorentz transformation will probably mix up ρ and $\underline{J}$.

Suppose $J^\mu = (\rho c, \underline{J})$ is a 4-vector. Let's see if (5.2) can be written covariantly as

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \quad (\text{scalar}) \tag{11.129}$$

$$\frac{\partial J^\mu}{\partial x^\mu} \equiv \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3}$$

$$= \frac{\partial(\rho c)}{\partial(ct)} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0 \tag{5.2}$$

So there are no relativistic corrections to (5.2). It can be written covariantly as (11.129).
If J^μ is a 4-vector, it must (by definition) transform like dx^μ under a Lorentz transformation. This tells us how ρ, $\underline{J}$ transform under a simple Lorentz transformation ($\underline{V} = V\hat{x}$): $(c\rho, \underline{J})$ must transform like $(cdt, d\underline{x})$.

dx^μ (Lorentz T.)	J^μ (Lorentz T.)	$\rho, \underline{J}$ (Galilean)
$cdt' = \gamma(cdt - \beta dx)$	$c\rho' = \gamma(c\rho - \beta J_x)$ (x11.53)	$\rho' = \rho$
$dx' = \gamma(dx - \beta c dt)$	$J_x' = \gamma(J_x - \beta c\rho)$ (x11.54)	$J_x' = J_x - \rho V$
$dy' = dy$	$J_y' = J_y$ (x11.55)	$J_y' = J_y$
$dz' = dz$	$J_z' = J_z$ (x11.56)	$J_z' = J_z$

Are these equations plausible?

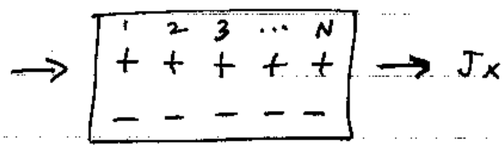
The last two (J_y, J_z) are identical to the Galilean transformation – no problem there.

The J_x transformation (XI1.54) differs from the Galilean result only by the factor γ, which $\to 1$ to 1^{st} order in v/c for $v/c \ll 1$. This can be understood in terms of time dilation (XI1.37, L28). Current density is charge / unit area / unit time. The unit area $dy\,dz$ doesn't change under a Lorentz trans. in the $\hat{x}$ direction, but the unit time does, by $1/\gamma$, so $J_x' \propto \gamma$. So far, so good.

But the ρ transformation (XI1.53) is wierd. Not only does it have the γ factor (which can be understood in terms of the Lorentz contraction of the volume element, since ρ = charge / unit volume), but it also has $\rho' \propto \rho - \beta J_x / c$, so $\rho = 0$ does not imply $\rho' = 0$!
(if $J_x \neq 0$). How can that be? Either a volume element contains charge, or it doesn't. How can it be charged in one frame but not in another? That's ridiculous!
(or maybe not. Einstein was pretty smart.)

let's look more closely –

Consider a box at rest in the unprimed frame, containing equal numbers of positive and negative charges ($\rho = 0$), but carrying a current $J_x \neq 0$. Suppose the ($-$) charges are stationary in the unprimed frame and the (+) charges move to the right:



Suppose the box contains N particles (+ and $-$) per unit yz area. When (+) particle #N leaves the box on the right, another (+) particle #0 enters on the left (to keep $\rho = 0$). The entry of #0 and the exit of #N are simultaneous events in the unprimed frame: $t_{N,out} = t_{0,in}$.
However, if we transform to another frame moving at $\underline{V} = V\hat{x}$, these events become non-simultaneous:

$$t'_{N,out} = \gamma\left(t_{N,out} - V x_N / c^2\right)$$

$$t'_{0,in} = \gamma\left(t_{0,in} - V x_0 / c^2\right)$$

$$t'_{0,in} - t'_{N,out} = \gamma V (x_N - x_0)/c^2 \neq 0 .$$

In the primed frame, there is a finite time interval after #N has left, but before #0 has entered. Primed observer thinks the box is negatively charged (agreeing with (XII.53).

So maybe Einstein was right. If we work it out in quantitative detail, this argument gives exactly (XII.53).

So much for charge conservation. How about Maxwell's equations? In addition to ρ and $\underline{J}$ (which we have combined into the 4-vector $J^\mu = (\rho c, \underline{J})$), the M.E.'s also contain the fields $\underline{E}$ and $\underline{B}$. (We will use the vacuum form, written in terms of $\underline{E}$ and $\underline{B}$, instead of the macroscopic forms in terms of $\underline{D}$ and $\underline{H}$. The macroscopic equations are not expected to be Lorentz invariant because they imply a preferred reference frame, the frame of the ponderable matter that produces $\underline{D}$ from $\underline{E}$ and $\underline{H}$ from $\underline{B}$.)

The fields $\underline{E}$ and $\underline{B}$ together have six components. That is too many for one 4-vector but not enough for two 4-vectors. In any case, the Galilean transformation $\underline{E}' = \underline{E} + \underline{v} \times \underline{B}/c$ already mixes $\underline{E}$ and $\underline{B}$, so we can expect that a Lorentz transformation also mixes them. That suggests combining them into a 2nd rank tensor. A general 2nd rank tensor has 16 ($= 4^2$) elements (too many). A symmetric 2nd rank tensor has only 10 independent elements (still too many). But an antisymmetric 2nd rank tensor has only six, which is just the right number. (All diagonal elements must be 0.)

So let's assume $\underline{E}$ and $\underline{B}$ are parts of an antisymmetric 2^{nd} rank tensor $F^{\mu\nu}$ (elements to be determined).
Since we already have ρ and $\underline{J}$ in terms of $J^{\mu} = (\rho c, \underline{J})$, let's start with the inhomogeneous M.E.'s (which contain ρ and $\underline{J}$):

$$\nabla \cdot \underline{E} = 4\pi\rho \qquad (XII.1)$$

$$\nabla \times \underline{B} - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\underline{J} \qquad (XII.2)$$

The RHS of (XII.1) is the 0^{th} component of J^{μ}, and the RHS of (XII.2) is the (1-3) components, so let's try combining both in the form

$$\frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = \frac{4\pi}{c} J^{\nu} \qquad (11.141)$$

The $\nu = 0$ component is $\frac{\partial F^{\mu 0}}{\partial x^{\mu}} = \frac{4\pi}{c} J^{0} = 4\pi\rho$, or

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = 4\pi\rho$$

To make this consistent with (XII.1), define $F^{00} = 0$ (already required by antisymmetry) and $F^{10} = E_x$, $F^{20} = E_y$, $F^{30} = E_z$. Thus we have, so far,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & & \\ E_y & & 0 & \\ E_z & & & 0 \end{pmatrix}$$

The $\nu = 1$ component of (11.141) is

$$\frac{\partial F^{\mu 1}}{\partial x^\mu} = \frac{4\pi}{c} J^1 = \frac{4\pi}{c} J_x$$

But the x-component of (x11.2) is

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c}\frac{\partial E_x}{\partial t} = \frac{4\pi}{c} J_x$$

So consistency requires $F^{01} = -E_x$ (already knew that) and $F^{11} = 0$ (already knew that), but also $F^{21} = B_z$, $F^{31} = -B_y$.
Similarly, the $\nu = 2$ or 3 component of (11.141), in comparison with the y- or z-component of (x11.2), gives $F^{32} = B_x$. We now have all the elements:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and the manifestly covariant equation

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad (11.141)$$

contains both of the inhomogeneous Maxwell eqns.

What about the homogeneous M.E.'s? Next time.