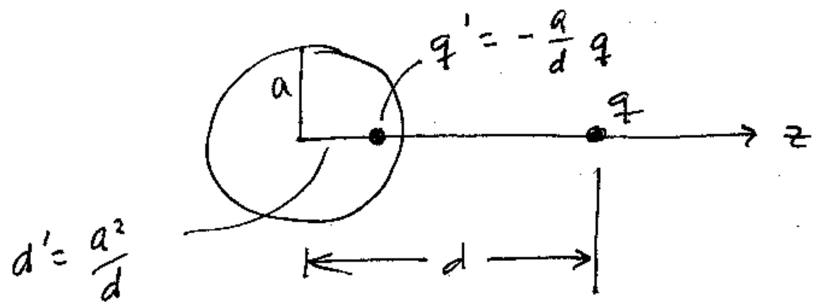
PHYS 532 L3

<u>More on image charges.</u>

Last time we derived the potential for a point charge q and a grounded conducting sphere of radius a:



$$\Phi(\underline{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\underline{x} - d\hat{z}|} - \frac{a/d}{|\underline{x} - \frac{a^2}{d}\hat{z}|}\right]$$

We can use this result to derive the potential around a conducting sphere in the presence of a uniform background field $\underline{E}_0$.

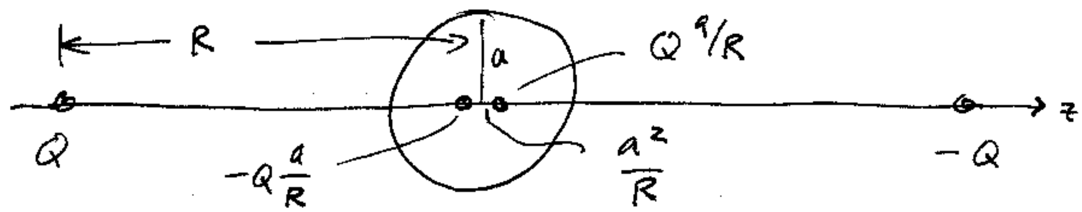
First consider two charges $\mp Q$ located at $z = \pm R$. The field at the origin is

$$\underline{E}(0) = \frac{2Q}{4\pi\epsilon_0 R^2}\hat{z}$$

If we let $R \to \infty$ but keep $Q/R^2 = \text{const}$, the field near the origin will be uniform, and

$$\Phi = -E_0 z \quad \text{with} \quad E_0 = \frac{2Q}{4\pi\epsilon_0 R^2}$$

Now add the potential from the two image charges using the previous result —



$$\Phi = -E_0 z + \frac{Q\,a/R}{4\pi\epsilon_0 \left|\underline{x} - \frac{a^2}{R}\hat{z}\right|} - \frac{Q\,a/R}{4\pi\epsilon_0\left|\underline{x} + \frac{a^2}{R}\hat{z}\right|}$$

from $\mp Q$ at $\pm R$

from image charge at $z = +a^2/R$

from image chg at $z = -a^2/R$

Now

$$\left|\underline{x} \mp \frac{a^2}{R}\hat{z}\right| = \sqrt{x^2+y^2+\left(z \mp \frac{a^2}{R}\right)^2}$$

$$\approx \sqrt{x^2+y^2+z^2 \mp \frac{2a^2 z}{R}}$$

$$= r\left(1 \mp \frac{2a^2 z}{r^2 R}\right)^{1/2}$$

$$\approx r \mp \frac{a^2 z}{rR} = r\left(1 \mp \frac{a^2 z}{r^2 R}\right)$$

$$\Rightarrow \quad \Phi \approx -E_0 z + \frac{Qa}{4\pi\epsilon_0 R r}\left[\frac{1}{1-\frac{a^2 z}{r^2 R}} - \frac{1}{1+\frac{a^2 z}{r^2 R}}\right]$$

$$\approx -E_0 z + \frac{Qa}{4\pi\epsilon_0 R r}\,\frac{2a^2 z}{r^2 R}$$

$$= -E_0 z + \frac{Q}{2\pi\epsilon_0 R^2}\,\frac{a^3 z}{r^3}$$

$\equiv E_0$

$$\text{so} \quad \Phi \approx -E_0 z + \frac{E_0 a^3 z}{r^3}$$

$$= -E_0 z \left(1 - \frac{a^3}{r^3}\right) \qquad (2.14)$$

$\to 0$ at $r = a$.

The perturbation potential added by the conducting sphere is a <u>dipole</u> potential. (see below).

End of image-charge examples.
We will skip the formal solution of boundary-value problems in electrostatics (rest of chapter 2, all of ch. 3).

<u>Potential of a Localized Charge Distribution</u>:

Recall that a charge-density distribution $\rho(\underline{x}')$ produces a potential

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d^3x' \qquad (1.17)$$

Suppose $\rho(\underline{x}')$ is localized near the origin, and the observer is far way ($|\underline{x}| \gg$ source size).

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{\sqrt{|\underline{x}|^2 - 2\underline{x}\cdot\underline{x}' + |\underline{x}'|^2}}$$

$$= \frac{1}{|\underline{x}| \sqrt{1 - \frac{2\hat{n}\cdot\underline{x}'}{|\underline{x}|} + \frac{|\underline{x}'|^2}{|\underline{x}|^2}}}$$

where $\hat{n} \equiv \frac{\underline{x}}{|\underline{x}|}$

Expand this with Taylor's theorem —

$$\frac{1}{|\underline{x}-\underline{x}'|} \approx \frac{1}{|\underline{x}|}\left[1 - \frac{1}{2}\left(-2\frac{\hat{n}\cdot\underline{x}'}{|x|} + \frac{|\underline{x}'|^2}{|\underline{x}|^2}\right) + \frac{3}{8}\left(\quad \uparrow \quad \right)^2 - + \cdots\right]$$

Keep terms to second order in $\frac{|\underline{x}'|}{|\underline{x}|}$ —

$$\frac{1}{|\underline{x}-\underline{x}'|} \approx \frac{1}{|\underline{x}|} + \frac{\hat{n}\cdot\underline{x}'}{|x|^2} + \frac{|\underline{x}'|^2}{|\underline{x}|^3}\left[-\frac{1}{2} + \frac{3}{2}\left(\frac{\hat{n}\cdot\underline{x}'}{|\underline{x}'|}\right)^2\right]$$

Plugging this into (1.17) gives

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\underline{x}|} + \frac{\underline{p}\cdot\hat{n}}{|\underline{x}|^2} + \frac{1}{2}\frac{\hat{n}\cdot\underline{\underline{Q}}\cdot\hat{n}}{|\underline{x}|^3}\right] \quad (\sim 4.10)$$

where

$$q = \int \rho(\underline{x}')\, d^3x' = \text{total charge (scalar)}$$

$$\underline{p} = \int \rho(\underline{x}')\,\underline{x}'\, d^3x' = \text{dipole moment (vector)} \quad (4.8)$$

$$\underline{\underline{Q}} = \int d^3x'\, \rho(\underline{x}')\left(3\,\underline{x}'\underline{x}' - \underline{\underline{1}}\,|\underline{x}'|^2\right) \quad (4.9)$$

$= \text{quadrupole moment } (2^{\text{nd}} \text{ rank tensor}).$

or, in index notation,

$$p_i = \int \rho(\underline{x}')\, x_i'\, d^3x'$$

$$Q_{ij} = \int d^3x'\, \rho(\underline{x}')\left(3x_i' x_j' - |\underline{x}'|^2\,\delta_{ij}\right)$$

$$\delta_{ij} = \text{Kronecker delta} = \begin{Bmatrix} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{Bmatrix}$$

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\underline{x}|} + \frac{p_i n_i}{|\underline{x}|^2} + \frac{1}{2}\frac{n_i Q_{ij} n_j}{|\underline{x}|^3} + \cdots\right] \qquad (4.10)$$

Einstein summation convention: If an index appears twice in a product, sum over that index.

$$\underline{p}\cdot\hat{n} \equiv p_i n_i \equiv p_1 n_1 + p_2 n_2 + p_3 n_3$$

$$\begin{aligned}\hat{n}\cdot\underline{\underline{Q}}\cdot\hat{n} \equiv\ & n_1 Q_{11} n_1 + n_1 Q_{12} n_2 + n_1 Q_{13} n_3 \\ & + n_2 Q_{21} n_1 + n_2 Q_{22} n_2 + n_2 Q_{23} n_3 \\ & + n_3 Q_{31} n_1 + n_3 Q_{32} n_2 + n_3 Q_{33} n_3 \\ \equiv\ & n_i Q_{ij} n_j\end{aligned}$$

Jackson 4.3 : Elementary Treatment of Electrostatics with Ponderable Media

Within a dielectric, at atomic length scales, the microscopic electric field $\underline{e}$ can be quite large:

$$|\underline{e}| \sim \frac{e}{4\pi\epsilon_0 a_0^2} \sim \frac{1.6\times10^{-19}\,C}{4\pi\times(8.85\times10^{-12})(0.5\times10^{-10}\,m)^2}$$

$$\sim 6\times10^{11}\ V/m = 6\times10^{8}\ V/mm$$

Bohr radius

When we apply an electric field to a dielectric, we are talking about a relatively tiny macroscopic field $E \lesssim 10^2$ V/mm, superimposed on the very strong small-scale fields that hold the dielectric together.

Nevertheless, even the wimpy macroscopic fields can significantly affect the dielectric medium, in one of two ways:

- It may distort the electron cloud of an atom, producing a small electric dipole moment where none existed before,
- If the molecules have permanent, but randomly oriented, dipole moments, the external field tends to make them less randomly oriented.

Either way, we can define a dipole moment density per unit volume by

$$\underline{P} = \sum_s \underbrace{N_s}_{\text{number density of species } s} \langle \underline{p}_s \rangle \qquad (X\,4.1)$$

The multipole expansion (4.10) can be written

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\underline{p}\cdot\underline{x}}{r^3} + \frac{1}{2}\frac{\underline{x}\cdot\underline{\underline{Q}}\cdot\underline{x}}{r^5} + \cdots\right]$$

where $\underline{x}$ = vector from source point to observing point.

Thus the potential at $\underline{x}$ from a dipole $\underline{p}$ at $\underline{x}'$ is

$$\Delta \Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \frac{\underline{p}\cdot(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}$$

or, from a continuous distribution $\underline{P}(\underline{x}')$

$$\Delta \Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\underline{P}(\underline{x}')\cdot(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}$$

The integral has significant contributions from distant regions of the dielectric, because $\int d^3x' \, \frac{\underline{P}\cdot(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3} \sim \int |\underline{P}| \, dr'$. The same is not true for the quadrupole term, which is $\sim \int d^3x' \, \frac{Q}{|r'|^3} \sim \int Q \, \frac{dr'}{r'}$, and even less true of higher-order terms. The quadrupole and higher-order terms are treated as "near field" and will be discussed in the next lecture.

Since $\nabla' \frac{1}{|\underline{x}-\underline{x}'|} = \frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}$, we have

$$\Delta \Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \underline{P}(\underline{x}')\cdot\nabla' \frac{1}{|\underline{x}-\underline{x}'|} .$$