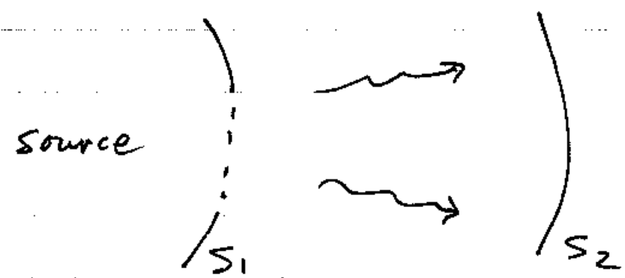
The scalar Kirchhoff diffraction theory (last time) gives the formula

$$\psi(\underline{x}) = \frac{-1}{4\pi}\int_{S_1} \frac{e^{ikR}}{R}\, \hat{n}'\cdot\left[\nabla'\psi + \left(ik - \frac{1}{R}\right)\frac{\underline{R}}{R}\psi\right] da' \quad (10.79)$$

for the field components $\psi(\underline{x})$ between a diffraction screen S_1 and a distant observing screen S_2



This is the solution of the Helmholtz eqn. with boundary conditions:

(1) ψ and $\frac{\partial\psi}{\partial n}$ vanish on S_1 except in the apertures.

(2) In the apertures, ψ and $\frac{\partial\psi}{\partial n}$ have the incident-wave values.

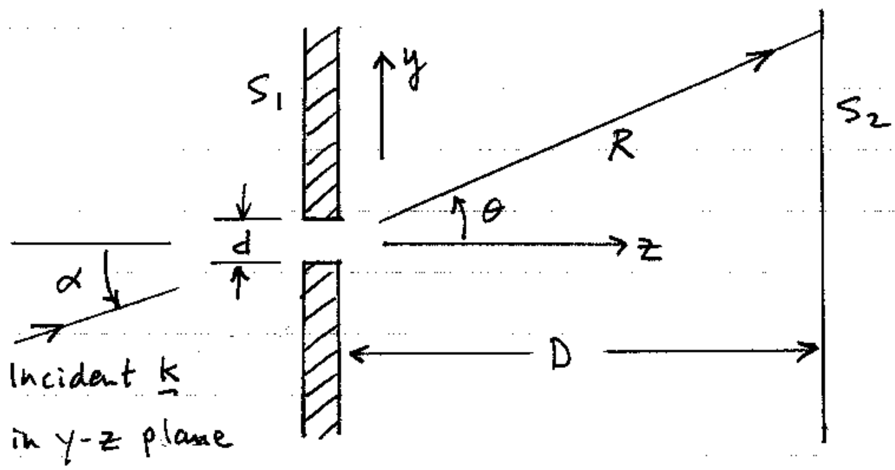
(3) On S_2, $\psi \to e^{ikR}/R$.

The approximation is reasonable when the size of the apertures is >> wavelength. However, it is logically inconsistent because specifying both ψ and $\frac{\partial\psi}{\partial n}$ on S_1 is overspecifying the problem mathematically.

This inconsistency can be fixed by using either Dirichlet or Neumann Green's functions instead of e^{ikR}/R. We won't do that here. It doesn't change the answer very much. Overspecifying is ok if you do so accurately, consistent with governing equation.

Let's work out a simple example –

Single-slit Fraunhofer diffraction



Slit is infinitely long in x direction ($\perp$ page)
$\rightarrow$ Nothing depends on x.

Incident wave has $\psi = \psi_0 e^{ik(\sin\alpha\, y + \cos\alpha\, z)}$

Diffraction theory requires $kd \gg 1$.

Kirchhoff approximation gives the boundary cond's on S_1:

$$\psi(x,y,0) = 0, \quad \frac{\partial\psi}{\partial n}(x,y,0) = 0 \qquad \text{for } |y| > d/2$$

$$\psi(x,y,0) = \psi_0 e^{ik\sin\alpha\, y}, \quad \frac{\partial\psi}{\partial n}(x,y,0) = ik\cos\alpha\, \psi_0 e^{ik\sin\alpha\, y}$$

for $|y| < d/2$

Since nothing depends on x, we can put the observation point at $x=0$ with no loss of generality. Then (10.79) gives

$$\psi(0,y,D) = -\frac{\psi_0}{4\pi}\int_{-d/2}^{d/2} dy' \int_{-\infty}^{\infty} dx' \frac{e^{ikR}}{R}\left[ik\cos\alpha + ik\left(1+\frac{i}{kR}\right)\cos\theta\right] e^{ik\sin\alpha\, y'}$$

Now neglect $\frac{i}{kR}$ compared to 1, and define $\mathcal{R} \equiv \sqrt{y^2+D^2}$. Then

$$R = \sqrt{x'^2 + (y-y')^2 + D^2} \approx \mathcal{R}\left[1 + \frac{x'^2}{2\mathcal{R}^2} + \frac{y'^2}{2\mathcal{R}^2} - \frac{yy'}{\mathcal{R}^2}\right]$$

assuming x', y', and y all $\ll D$.

$$\Rightarrow \psi(0,y,D) \approx \frac{-\psi_0}{4\pi}\frac{e^{ik\mathcal{R}}}{\mathcal{R}}\, ik(\cos\alpha + \cos\theta)\int_{-d/2}^{d/2} dy' \int_{-\infty}^{\infty} dx'\; e^{ik\left[\frac{x'^2}{2\mathcal{R}} + \frac{y'^2}{2\mathcal{R}} - \frac{yy'}{\mathcal{R}} + y'\sin\alpha\right]} \qquad (X10.11)$$

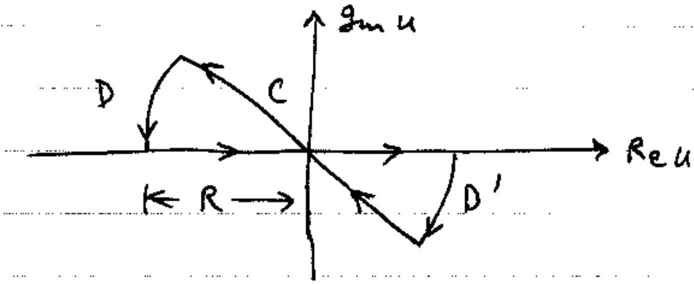
First work on the x' integration. <u>Define</u>

$$u = e^{i\frac{3\pi}{4}}\sqrt{\frac{k}{2\mathcal{R}}}\, x'$$

$$\Rightarrow dx' = \sqrt{\frac{2\mathcal{R}}{k}}\, e^{-i\frac{3\pi}{4}} du \quad \text{and} \quad u^2 = e^{i\frac{3\pi}{2}}\frac{kx'^2}{2\mathcal{R}} = -i\frac{kx'^2}{2\mathcal{R}}$$

$$\Rightarrow I \equiv \int_{-\infty}^{\infty} dx'\, e^{ikx'^2/2\mathcal{R}} = \sqrt{\frac{2\mathcal{R}}{k}}\, e^{-i\frac{3\pi}{4}}\int_C e^{-u^2} du$$

The contour C for the u integral is along the line $u = e^{i\,3\pi/4}\,x'$ where x' varies from $-\infty$ to ∞:



Closing this contour by joining it with D, D', and the real u axis, and using the fact that e^{-u^2} is analytic within this contour, we have

$$\int_{D'} e^{-u^2}du + \int_C e^{-u^2}du + \int_D e^{-u^2}du + \int_{-R}^{R} e^{-u^2}du = 0$$

as $R \to \infty$, the integrals along D' and D vanish, so

$$\int_C e^{-u^2}du = -\int_{-\infty}^{\infty} e^{-u^2}du = -\sqrt{\pi}$$

so $$I = -\sqrt{\frac{2\pi R}{k}}\, e^{-i\frac{3\pi}{4}} = e^{i\pi/4}\sqrt{\frac{2\pi R}{k}} \qquad (x10.12)$$

Now for the y' integration. Here we need another approximation. <u>Assume</u>

$$\frac{k\,d^2}{2D} \ll 1$$

This case is called "Fraunhofer diffraction".

(The opposite case is called "Fresnel diffraction.")

Then, since $y'^2 \le d^2$ and $R \sim D$, the term $\frac{y'^2}{2R}$ in the exponent of (X10.11) can be neglected compared to the other two terms which are $\propto y'$:

$$\frac{y'^2}{2R} - \frac{yy'}{R} + y'\sin\alpha \approx y'\left(\sin\alpha - \frac{y}{R}\right)$$

$$\approx y'(\sin\alpha - \sin\theta)$$

since $\sin\theta \equiv \frac{y}{D} \approx \frac{y}{R}$

Then the y' integration becomes trivial :

$$\int_{-d/2}^{d/2} dy'\, e^{iky'(\sin\alpha-\sin\theta)} = \frac{e^{\frac{ikd}{2}(\sin\alpha-\sin\theta)} - e^{-\frac{ikd}{2}(\sin\alpha-\sin\theta)}}{ik(\sin\alpha-\sin\theta)}$$

$$\equiv d\,\frac{\sin\left[\frac{kd}{2}(\sin\alpha-\sin\theta)\right]}{\left[\frac{kd}{2}(\sin\alpha-\sin\theta)\right]} \qquad (X10.13)$$

Finally (!), substituting (X10.12) and (X10.13) into (X10.11) gives

$$\psi = \frac{-ik\psi_0}{4\pi}\,\frac{e^{ikR}}{R}\,(\cos\alpha+\cos\theta)\,e^{i\pi/4}\sqrt{\frac{2\pi R}{k}}\,d\,\frac{\sin\left[\frac{kd}{2}(\sin\alpha-\sin\theta)\right]}{\left[\frac{kd}{2}(\sin\alpha-\sin\theta)\right]}$$

$$= -\psi_0\, e^{i\,3\pi/4}\sqrt{\frac{kd^2}{2\pi R}}\; e^{ikR}\,\frac{\cos\alpha+\cos\theta}{2}\;\frac{\sin\left[\frac{kd}{2}(\sin\theta-\sin\alpha)\right]}{\left[\frac{kd}{2}(\sin\theta-\sin\alpha)\right]}$$

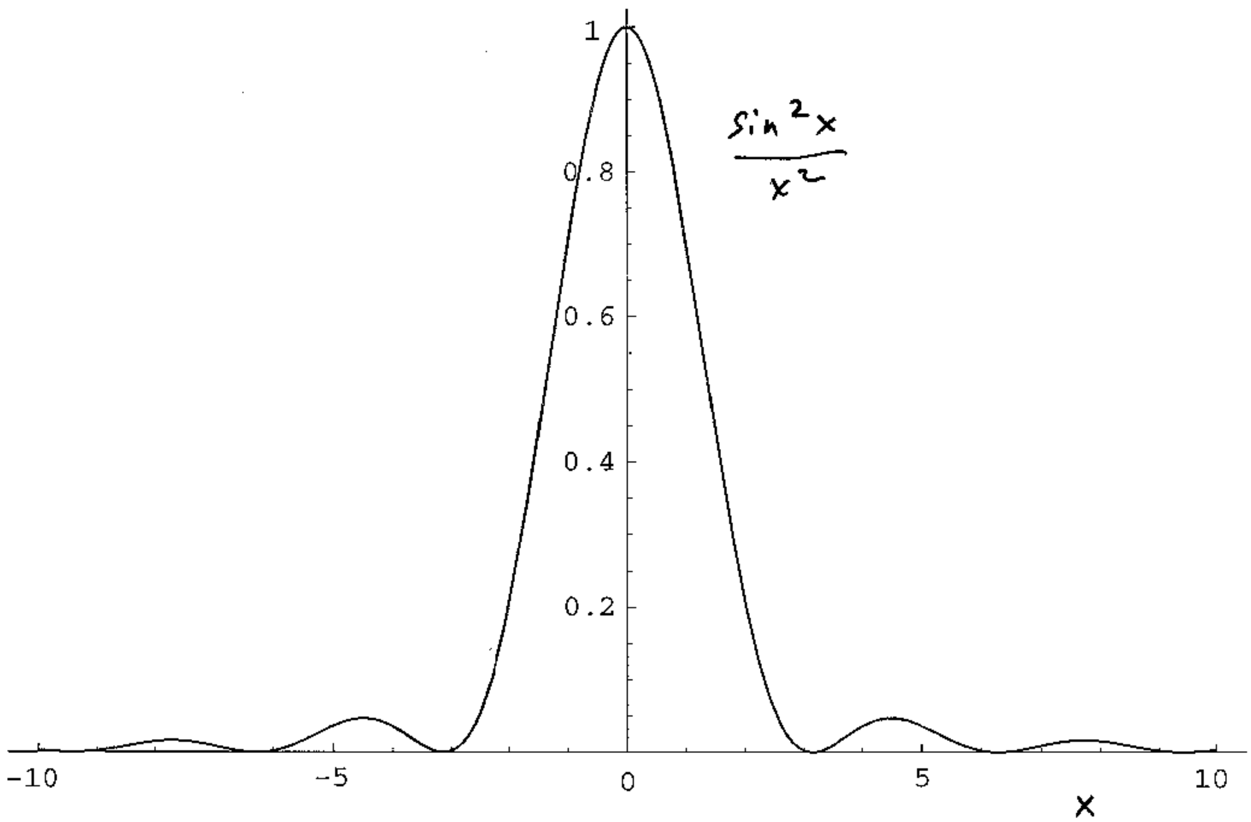
(X10.14)

The intensity $I \propto |\psi|^2$ so

$$\boxed{I(\theta) \propto \frac{kd^2}{2\pi R} |\psi_0|^2 \left(\frac{\cos\alpha + \cos\theta}{2}\right)^2 \frac{\sin^2\left[\frac{kd}{2}(\sin\theta - \sin\alpha)\right]}{\left[\frac{kd}{2}(\sin\theta - \sin\alpha)\right]^2}}$$

(X 10.15)

The function $\frac{\sin^2 x}{x^2}$ has a peak ($=1$) at $x = 0$:



The width of the peak is equal to the x-value of the first zero : $x = \pi$ or

$$\frac{kd}{2}(\sin\theta - \sin\alpha) = \pi$$

$$\sin\theta - \sin\alpha = 2\sin\frac{\theta-\alpha}{2}\cos\frac{\theta+\alpha}{2} \approx (\theta-\alpha)\cos\alpha$$

$\rightarrow$ Peak width $\Delta\theta = \dfrac{2\pi}{kd\cos\alpha} \ll 1$

(unless $\cos\alpha \lesssim \dfrac{1}{kd} \ll 1$).

For $\alpha = 0$ (normal incidence), $I \propto \dfrac{\sin^2\left(\frac{kd}{2}\sin\theta\right)}{\left(\frac{kd}{2}\sin\theta\right)^2}$

$\rightarrow 0$ at $\frac{kd}{2}\sin\theta = n\pi$, $\sin\theta = n\lambda/d$

Geometric interpretation: