The scalar Kirchhoff diffraction theory (last time) gives the formula

$$\Psi(x) = \frac{1}{4\pi} \int \frac{e^{ikR}}{R} \hat{n}' \left[ \nabla' \Psi + (ik - \frac{i}{R}) \frac{R}{k} \Psi \right] \, d\alpha' \quad (10.79)$$

for the field components $\Psi(x)$ between a diffraction screen $S_1$ and a distant observing screen $S_2$.

```
        \text{Source}

    \quad \xrightarrow{\text{Source}} \xrightarrow{\text{R}} 

    / \quad / \\
S_1 \quad S_2
```

This is the solution of the Helmholtz eqn. with boundary conditions:

1. $\Psi$ and $\frac{\partial \Psi}{\partial n}$ vanish on $S_1$ except in the apertures.

2. In the apertures, $\Psi$ and $\frac{\partial \Psi}{\partial n}$ have the incident-wave values.

3. On $S_2$, $\Psi \to e^{ikR/R}$.

The approximation is reasonable when the size of the apertures is $\gg$ wavelength. However, it is logically inconsistent because specifying both $\Psi$ and $\frac{\partial \Psi}{\partial n}$ on $S_1$, is overspecifying the problem mathematically.
This inconsistency can be fixed by using either Dirichlet or Neumann Green's functions instead of $e^{i k R / R}$. We won't do that here. It doesn't change the answer very much. Overspecifying is OK if you do so accurately, consistent with governing equation.

Let's work out a simple example -

Single-slit Fraunhofer diffraction

\[
\begin{align*}
\text{Incident wave has } &\psi = \psi_0 e^{i k (\sin \theta + \cos \theta z)} \\
\text{Diffraction theory requires } &kd \gg 1.
\end{align*}
\]

Kirchhoff approximation gives the boundary conditions on $S_1$:

\[
\begin{align*}
\psi(x, y, 0) &= 0, & \frac{\partial \psi}{\partial n}(x, y, 0) &= 0 \quad \text{for } |y| > d/2 \\
\psi(x, y, 0) &= \psi_0 e^{i k \sin \theta y}, & \frac{\partial \psi}{\partial n}(x, y, 0) &= i k \cos \theta \psi_0 e^{i k \sin \theta y} \quad \text{for } |y| < d/2
\end{align*}
\]
Since nothing depends on \( x \), we can put the observation point at \( x = 0 \) with no loss of generality. Then (10.74) gives

\[
\Psi(0, y, D) = -\frac{\psi_0}{4\pi} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx' \frac{e^{ikR}}{R} \left[ i k \cos x + i k (1 + \frac{i}{KR}) \cos \phi \right] e^{i k x' y'} e^{-\frac{1}{2R} \left( \frac{x'^2}{2R} + \frac{y'^2}{R^2} \right)}
\]

Now neglect \( \frac{i}{KR} \) compared to 1, and define \( R = \sqrt{y^2 + D^2} \).

Then

\[
R = \sqrt{x'^2 + (y-y')^2 + D^2} = R \left[ 1 + \frac{x'^2}{2R^2} + \frac{y'^2}{2R^2} - \frac{y'^2}{R^2} \right]
\]

assuming \( x', y' \), and \( y \) all \( \ll D \).

\[
\Psi(0, y, D) = -\frac{\psi_0}{4\pi} \frac{e^{ikR}}{R} ik(\cos x + \cos \phi) \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx' \frac{e^{ik \left[ \frac{x'^2}{2R} + \frac{y'^2}{2R} - \frac{y'^2}{R^2} + y' \sin \phi \right]}}{R}
\]

\[
(\times 10.11)
\]

First work on the \( x' \) integration. Define

\[
u = e^{i \frac{3\pi}{4}} \sqrt{\frac{k}{2R}} x'
\]

\[
dx' = \sqrt{\frac{2R}{k}} e^{-i \frac{3\pi}{4}} dy' \quad \text{and} \quad u^2 = e^{i \frac{3\pi}{2}} \frac{k x'^2}{2R}
\]

\[
e^{-i \frac{k x'^2}{2R}}
\]

\[
I = \int_{-\infty}^{\infty} dx' e^{ikx'^2/2R} = \sqrt{\frac{2R}{k}} e^{-i \frac{3\pi}{4}} \int_{-\infty}^{\infty} e^{-u^2} du
\]
The contour \( C \) for the \( u \) integral is along the line \( u = e^{-i \frac{3\pi}{4}} x' \) where \( x' \) varies from \(-\infty \) to \( \infty \)

\[
\int_{D'} e^{-u^2} du + \int_{C} e^{-u^2} du + \int_{D} e^{-u^2} du + \int_{-R} e^{-u^2} du = 0
\]

As \( R \to \infty \), the integrals along \( D' \) and \( D \) vanish, so

\[
\int_{C} e^{-u^2} du = -\int_{-\infty}^{0} e^{-u^2} du = -\sqrt{\pi}
\]

so \( I = -\sqrt{\pi \frac{2\pi R}{k}} e^{-i \frac{3\pi}{4}} = e^{i \frac{3\pi}{4}} \sqrt{\frac{2\pi R}{k}} \) \((x10.12)\)

Now for the \( y' \) integration. Here we need another approximation. Assume

\[
\frac{k d^2}{2D} \ll 1
\]

This case is called "Fraunhofer diffraction".

(The opposite case is called "Fresnel diffraction.")
Then, since \( y'^2 \leq d^2 \) and \( R \approx D \), the term \( \frac{y'^2}{2R} \) in the exponent of \((X.10.11)\) can be neglected compared to the other two terms which are \( \propto y' \):

\[
\frac{y'^2}{2R} - \frac{yy'}{R} + y' \sin \theta \propto y' \left( \sin \theta - \frac{y}{R} \right)
\]

\[
\approx y' \left( \sin \theta - \sin \theta \right)
\]

Since \( \sin \theta \approx \frac{y}{d} \approx \frac{y}{R} \)

Then the \( y' \) integration becomes trivial:

\[
\int_{-d/2}^{d/2} dy' e^{iky' \left( \sin \theta - \sin \theta \right)} = \frac{e^{ikd/2 \left( \sin \theta - \sin \theta \right)} - e^{-ikd/2 \left( \sin \theta - \sin \theta \right)}}{ik \left( \sin \theta - \sin \theta \right)}
\]

\[
\approx d \frac{\sin \left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}{\left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}
\]

\((X.10.13)\)

Finally \((5)\), substituting \((X.10.12)\) and \((X.10.13)\) into \((X.10.11)\) gives

\[
\psi = \frac{ikd_0}{4\pi} e^{ikR \left( \cos \theta + \cos \theta \right)} e^{ik \left( \frac{3}{16} \right)} d \frac{\sin \left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}{\left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}
\]

\[
= -4_0 e^{ik \frac{kd - L}{2\pi R}} e^{ikR \left( \cos \theta + \cos \theta \right)} \frac{\sin \left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}{\left[ \frac{kd}{2} \left( \sin \theta - \sin \theta \right) \right]}
\]

\((X.10.14)\)

The intensity, \( I \propto |\psi|^2 \), so
\[ I(\theta) \propto \frac{k d^2}{2 \pi R} \left[ \gamma_0 \right]^2 \left( \cos \phi + \cos \theta \right)^2 \frac{\sin^2 \left[ \frac{k d}{2} (\sin \phi - \sin \theta) \right]}{\left[ \frac{k d}{2} (\sin \phi - \sin \theta) \right]^2} \]

The function \( \frac{\sin^2 x}{x^2} \) has a peak \((= 1)\) at \( x = 0 \):

The width of the peak is equal to the \( x \)-value of the first zero: \( x = \pi \) or

\[ \frac{k d}{2} (\sin \theta - \sin \alpha) = \pi \]

\[ \sin \theta - \sin \alpha = 2 \sin \frac{\theta - \alpha}{2} \cos \frac{\theta + \alpha}{2} \approx (\theta - \alpha) \cos \alpha \]
Peak width \( \Delta \theta = \frac{2\pi}{kd \cos \alpha} \ll 1 \)

(unless \( \cos \alpha \ll \frac{1}{kd} \ll 1 \)).

For \( \alpha = 0 \) (normal incidence),
\[ I \propto \frac{\sin^2 \left( \frac{kd}{2} \sin \theta \right)}{(\frac{kd}{2} \sin \theta)^2} \]

\( \to 0 \) at \( \frac{kd}{2} \sin \theta = n\pi \), \( \sin \theta = n\lambda/kd \).

Geometric interpretation:

\[ ds \sin \theta = n\lambda \]