PHYS 532 L25

Scattering and Diffraction (Jackson §10.1, 10.5)

10.1 Scattering at long wavelengths

Assume a plane wave incident on a small scatterer ($ka \ll 1$). Assume it induces an electric dipole $\underline{p}$ and/or a magnetic dipole $\underline{m}$ in the scatterer. Scattered radiation is the dipole radiation from the scatterer.

Assume the incident radiation has

$$\left.\begin{aligned} \underline{E}_{inc} &= \hat{\epsilon}_0 E_0 e^{i(k\hat{n}_0 \cdot \underline{x} - \omega t)} \\ \text{Faraday} \rightarrow \underline{B}_{inc} &= \frac{1}{c}\hat{n}_0 \times \underline{E}_{inc} \end{aligned}\right\} \quad (10.1)$$

Where $\hat{n}_0$ is the propagation direction and $\hat{\epsilon}_0$ the polarization vector of the incident radiation. (Jackson omits the factor $e^{-i\omega t}$ as usual.)

From (9.19), the electric dipole radiation from the scatterer has

$$\underline{E}_{E1,sc} = \frac{k^2}{4\pi\epsilon_0 r} e^{i(kr-\omega t)} \hat{n}\times(\underline{p}\times\hat{n}) \qquad (x10.1)$$

$$\underline{B}_{E1,sc} = \frac{Z_0 k^2}{4\pi r} e^{i(kr-\omega t)} \hat{n}\times\underline{p} \qquad (x10.2)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ as before.

from (9.35), the magnetic dipole radiation from the scatterer has

$$\underline{E}_{M1,sc} = -Z_0 \frac{k^2}{4\pi r} e^{i(kr-\omega t)} \hat{n} \times \underline{m} \qquad (9.36)$$

$$\underline{B}_{M1,sc} = \frac{\mu_0 k^2}{4\pi r} e^{i(kr-\omega t)} \hat{n} \times (\underline{m} \times \hat{n}) \qquad (x10.3)$$

Combining these, the scattered radiation has

$$\left.\begin{aligned} \underline{E}_{sc} &= \frac{k^2}{4\pi\epsilon_0 r} e^{i(kr-\omega t)} \left[\hat{n} \times (\underline{p} \times \hat{n}) - \frac{1}{c} \hat{n} \times \underline{m} \right] \\ \underline{B}_{sc} &= \frac{1}{i\omega} \nabla \times \underline{E}_{sc} = \frac{1}{c} \hat{n} \times \underline{E}_{sc} \end{aligned}\right\} \quad (10.2)$$

Here $\hat{n}$ is the direction of the scattered radiation and r is distance from scatterer.
The properties of the scatterer are contained in $\underline{p}$, $\underline{m}$.

The <u>differential scattering cross section</u> is defined by

$$\frac{d\sigma}{d\Omega} \equiv \frac{\text{scattered power per unit solid angle}}{\text{incident power per unit area}}$$

$$= \frac{r^2 \langle \underline{S}_{sc} \rangle \cdot \hat{n}}{\langle |\underline{S}_{inc}| \rangle} \qquad (x10.4)$$

$$\langle |\underline{S}_{inc}| \rangle = \frac{1}{2\mu_0} \mathrm{Re} \left| \left(\underline{E}_{inc} \times \underline{B}_{inc} \right) \right|$$

$$= \frac{1}{2\mu_0} \frac{E_0^2}{c} = \frac{1}{2Z_0} E_0^2$$

$$\langle \underline{S}_{sc} \rangle \cdot \hat{n} = \frac{[\underline{E}_{sc}^* \times (\hat{n} \times \underline{E}_{sc})] \cdot \hat{n}}{2\mu_0 c} = \frac{1}{2Z_0} |\underline{E}_{sc}|^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = r^2 |\underline{E}_{sc}|^2 / E_0^2$$

or, using (10.2),

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \frac{|\hat{n} \times (\underline{p} \times \hat{n}) - \hat{n} \times \underline{m}/c|^2}{E_0^2} \qquad (\times 10.5)$$

The cross-section for scattered radiation with polarization vector $\hat{\epsilon}$ is

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right)_{\hat{\epsilon}} = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \frac{|\hat{\epsilon}^* \cdot [\hat{n} \times (\underline{p} \times \hat{n}) - \hat{n} \times \underline{m}/c]|^2}{E_0^2}} \qquad (\sim 10.4)$$

Note that, if the ratios p/E_0 and m/E_0 are independent of frequency, then $\boxed{\frac{d\sigma}{d\Omega} \propto k^4 \propto \omega^4}$

This is called Rayleigh's law. It works for any dipole scatterer with size $a \ll \lambda = 2\pi/k$. (If this condition is not met, the charge and current densities induced in the scatterer cannot be described by simple moments $\underline{p}, \underline{m} \propto E_0$.)

Rayleigh scattering is more effective for shorter wavelengths than for longer wavelengths. This explains why the sky is blue and sunsets are red.

The magnitude of the dipole moments $\underline{p}, \underline{m}$ is proportional to the volume ($\propto a^3$) of the scatterer. For example, dielectric spheres with radius a and dielectric constant ϵ (>1) have $\underline{m} = 0$ and

$$\underline{p} = 4\pi\epsilon_0 a^3 \left(\frac{\epsilon - 1}{\epsilon + 2}\right) E_0 \qquad (10.5)$$

while perfectly conducting spheres have

$$\underline{p} = 4\pi\epsilon_0 a^3 \underline{E}_{inc} \qquad (10.12)$$

$$\underline{m} = -\frac{2\pi}{\mu_0} a^3 \underline{B}_{inc} \qquad (10.13)$$

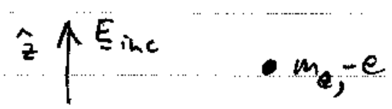
In both cases, (x10.5) implies

$$\frac{d\sigma}{d\Omega} \propto k^4 a^6 = (ka)^4 a^2$$

$$\propto (ka)^4 \times \text{geometric cross-section}.$$

See Jackson § 10.1.B for dielectric spheres, and § 10.1.C for perfectly conducting spheres.

Let's work out the differential scattering cross-section for Thompson scattering - scattering of long-wave radiation by a single free electron, mass m_e, charge $-e$.



$$m_e \ddot{z} = -eE \quad \rightarrow -m_e \omega^2 z = -eE$$

$$z = \frac{e}{m_e \omega^2} E_{inc}$$

$$p_z = -ez = \frac{-e^2}{m_e \omega^2} E_{inc} \qquad (x10.6)$$

from (10.4), $$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \frac{|n \times (\underline{p} \times \hat{n})|^2}{E_{inc}^2}$$

$$= \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \left(\frac{e^2}{m_e \omega^2}\right)^2 \sin^2\theta \qquad (x10.7)$$

(θ = angle between $\hat{z}$ and $\hat{n}$)

Setting $kc = \omega$ and defining the classical electron radius r_c by $\frac{e^2}{4\pi\epsilon_0 r_c} = m_e c^2$

($\rightarrow r_c = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \sim 3 \times 10^{-15}$ m), (10.7) becomes

$$\boxed{\frac{d\sigma}{d\Omega} = r_c^2 \sin^2\theta} \qquad (x10.8)$$

The total scattering cross section is

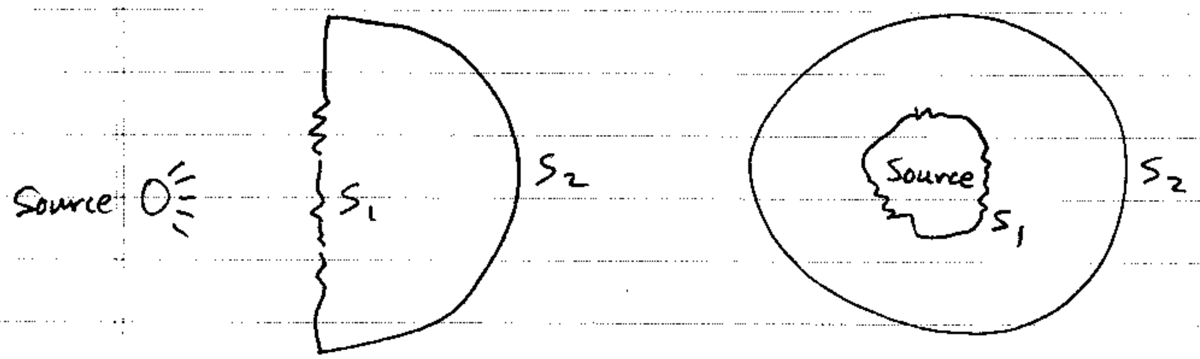
$$\sigma = r_c^2 \int_0^\pi \sin^2\theta \; 2\pi \sin\theta \, d\theta = 2\pi r_c^2 \left[2\int_0^1 (1-x^2)\,dx\right]$$

$$\sigma = \frac{8\pi}{3} r_c^2 \qquad (x10.9)$$

10.2 Diffraction

Diffraction through a screen with apertures is like the inverse of scattering, with each aperture acting as a source of scattered waves, but with one important difference: scattering theory assumes $a \ll \lambda$ while diffraction theory makes the opposite assumption $d \gg \lambda$ (d = aperture dimension). Waves are deflected by an angle $\theta \lesssim \lambda/d \ll 1$.

Assume a light source shining on a diffraction screen S_1 (opaque except for apertures), and the transmitted light is detected at a second screen S_2 far away. The shape of S_1 is arbitrary and the source region can be topologically open or closed:



The diffraction pattern is the intensity distribution at S_2. To lowest order, it is determined by geometrical optics (ray tracing). The next order of approximation involves diffraction at the apertures.

Scalar Kirchhoff Diffraction Theory

As usual, assume the scalar function ψ, representing any Cartesian component of $\underline{E}$ or $\underline{B}$, varies as $e^{-i\omega t}$ and its spatial part satisfies the Helmholtz equation $(\nabla^2 + k^2)\psi(\underline{x}) = 0$, (10.73) between S_1 and S_2.

If $\phi(\underline{x})$, $\psi(\underline{x})$ are arbitrary functions, the divergence theorem applied to the vector $\phi\nabla\psi - \psi\nabla\phi$ gives

$$\int_V \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)\, d^3x = \oint_S \hat{n}\cdot(\phi\nabla\psi - \psi\nabla\phi)\, da$$

$$= \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da$$

$\Rightarrow$ Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da \qquad (1.35)$$

Now, assume that ψ satisfies (10.73), and that $\phi(\underline{x}) = G(\underline{x}, \underline{x}')$, the corresponding Green's function that satisfies $(\nabla^2 + k^2)G(\underline{x}, \underline{x}') = -\delta^{(3)}(\underline{x} - \underline{x}')$. Then

$$\int_V \left[G(-k^2\psi) - \psi(-k^2 G - \delta^{(3)}(\underline{x} - \underline{x}')\right] d^3x'$$

$$= \oint_S \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) da = \psi(\underline{x})$$

In (1.35), S is a closed surface enclosing the volume V. Here, we take S to be the combination of S_1 and S_2. Defining $\hat{n}'$ as the inward normal of this surface, we have

$$\psi(\underline{x}) = \oint_S \left[\psi(\underline{x}')\,\hat{n}'\cdot\nabla' G(\underline{x},\underline{x}') - G(\underline{x},\underline{x}')\,\hat{n}'\cdot\nabla'\psi(\underline{x}') \right] da' \qquad (10.75)$$

Now set G = infinite-space Green's function for outgoing waves:

$$G(\underline{x}\,\underline{x}') = \frac{e^{ikR}}{4\pi R} \qquad (10.76)$$

Where $\underline{R} = \underline{x}-\underline{x}'$, $\quad \nabla' R = \frac{-\underline{R}}{R}$

$$\nabla' G = \left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{4\pi R}\,\nabla' R = \frac{-\underline{R}}{R}\left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{4\pi R} \qquad (X10.10)$$

Substituting (10.76) and (X10.10) in (10.75) gives

$$\psi(\underline{x}) = \frac{-1}{4\pi}\oint_S \frac{e^{ikR}}{R}\,\hat{n}'\cdot\left[\nabla'\psi + \left(ik - \frac{1}{R}\right)\frac{\underline{R}}{R}\,\psi\right] da' \qquad (10.77)$$

At surface S_2 the waves are outgoing:

$$\psi_{S_2} = f(\theta,\phi)\,\frac{e^{ikR}}{R}$$

$$\nabla'\psi = \left[\nabla' f + \nabla' R\left(ik - \frac{1}{R}\right) f\right]\frac{e^{ikR}}{R}$$

Since $f = f(\theta,\varphi)$, its gradiant is $\perp \underline{r}$ and does not contribute at S_2 (no component along $\hat{n}'$).

$$\nabla' \psi = \left(\frac{1}{R}\right) - \frac{\underline{R}}{R}\left(ik - \frac{1}{R}\right)\psi$$

$\therefore$ both ψ and $\nabla'\psi \to 0$ at least as fast as $1/R$. If we place S_2 far enough away, the surface integral over $S_2 \to 0$ and we have

$$\boxed{\psi(\underline{x}) = \frac{-1}{4\pi}\int_{S_1} \frac{e^{ikR}}{R}\,\hat{n}'\cdot\left[\nabla'\psi + \left(ik - \frac{1}{R}\right)\frac{\underline{R}}{R}\psi\right]da'} \qquad (10.79)$$

the Kirchhoff integral formula.

Fields are determined by an integral over the source surface S_1. (This makes sense.)