Vector Spherical Harmonic Theory of Radiation
(Jackson 9.6 - 9.12)

Until now, our discussion of radiation from a localized source has used a power-series expansion in \( k = \omega / c \), \( \hat{n} \) - unit vector from source to observer, and \( x' \) - position within source. This is useful for \( k a \ll 1 \) where \( a \) = source size.

We obtained expressions for the radiation wavefields (\( \propto 1/r \)) for an electric dipole source (the \( n = 0 \) term of the expansion) and for a magnetic dipole or electric quadrupole source (both from the \( n = 1 \) term of the expansion).

We could proceed to the \( n = 2, 3, \ldots \) terms and find wave fields for magnetic quadrupole, electric octupole, etc., sources. But the results would be too ugly to contemplate.

There is a better way, utilizing vector spherical harmonics. It is more difficult mathematically, but more "elegant" and more powerful than the power-series approach.

Unfortunately (\( \Box \)) we don't have time to cover this approach in any detail. What follows is just a superficial outline of the approach. Please read Jackson 9.6 - 9.12.
The source-free scalar wave equation is

\[ (\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}) \phi = 0 \]  

(9.77)

where the scalar field \( \phi \) could be the scalar potential \( \Phi \) or any cartesian component of \( A \).

If \( \phi \propto e^{-i\omega t} \), this reduces to the Helmholtz eqn.

\[ (\nabla^2 + k^2) \phi = 0 \]  

(\( k = \omega/c \))  

(9.79)

The general solution of (9.79) in spherical coordinates is

\[ \phi = \sum_{l,m} \left[ a_{lm} j_l(kr) + b_{lm} n_l(kr) \right] Y_{lm}(\theta, \phi) \]

where \( Y_{lm}(\theta, \phi) \) = spherical harmonic function (chap. 3).

\( j_l(kr) \) = Spherical Bessel fn. \( \propto \frac{J_{l+\frac{1}{2}}(kr)}{\sqrt{kr}} \)

\( n_l(kr) \) = Spherical Neumann fn. \( \propto \frac{N_{l+\frac{1}{2}}(kr)}{kr} \)

and \( J, N \) are ordinary Bessel, Neumann fn's.

(Solutions of Bessel's eqn.)

Jackson's series expansions (9.88) show that, for \( x \ll 1 \),

\[ j_l(x) \rightarrow x^l \left[ 1 + O(x^2) \right] \] (regular at origin)

while \( n_l(x) \rightarrow x^{-(l+1)} \left[ 1 + O(x^2) \right] \) (irregular at origin)
Define the spherical Hankel functions

\[ h_{l}^{(1)}(kr) = j_{l}(kr) + \alpha n_{l}(kr) \]

From Jackson's (9.84), for large \( kr \),

\[ h_{l}^{(1)}(kr) \sim (-1)^{l+1} \frac{e^{ikr}}{kr} \]

corresponding to an outgoing wave. \( h_{l}^{(2)} \) corresponds to an incoming wave converging at the origin, and is of no further interest.

The Green's function for the Helmholtz eqn. satisfies

\[ (\nabla^2 + k^2) G(x, x') = -\delta^{(3)}(x-x') \]

and the Green's fn. for an outgoing wave is

\[ G(x, x') = -\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \quad (9.94) \]

\[ = ik \sum_{l=0}^{\infty} j_{l}(kr_{2}) h_{l}^{(1)}(kr_{3}) \sum_{m=-l}^{l} Y_{lm}(\theta_{2}, \phi_{2}) Y_{lm}^{*}(\theta_{3}, \phi_{3}) \quad (9.98) \]

where \( r_{2}, r_{3} = \text{smaller, greater of } (r, r') \).

For \( r < r' \), this is a solution of the homogeneous wave eqn. that is regular at the origin.

For \( r > r' \), it is an outgoing wave.
The spherical harmonics $Y_{lm}$ are solutions of the angular part of the Helmholtz eqn.

\[ \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} Y_{lm} = 0 \quad (9.99) \]

which can be written in the quantum mechanical form

\[ L^2 Y_{lm} = \ell (\ell + 1) Y_{lm} \quad (9.100) \]

if we define the rotation operator $L$ such that

\[ \text{i} L = r \times \nabla \quad (9.101) \]

The cartesian components of $L$ are given by

\[ \text{i} L_x = -\sin \phi \frac{\partial}{\partial \theta} - \cos \phi \cot \theta \frac{\partial}{\partial \phi} \]

\[ \text{i} L_y = \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \]

\[ \text{i} L_z = \frac{\partial}{\partial \phi} \]

$iL_z$ represents rotation about the $z$ axis.

($\text{i} L = \text{orbital angular momentum operator of q.m.}$)

Can verify by direct substitution that $-L^2$ is the angular operator in (9.99), so that

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\ell^2}{r^2} \quad (9.106) \]
Note also that \( r \cdot l = 0 \), and that \( L_z Y_{lm} = m Y_{lm} \).

Using \( l \), we can define the vector spherical harmonics

\[
X_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm} \tag{9.119}
\]

which are vector functions associated with the scalar functions \( Y_{lm} \).

Why bother?

Because "it can be shown" (by Jackson) that the source-free Maxwell equations have the general solutions

\[
\begin{align*}
\vec{H} &= \sum_{l,m} \left[ \alpha_E (l,m) Y_{lm}^{(1)} (kr) \times \vec{X}_{lm} - \frac{i}{k} \alpha_m (l,m) \vec{\nabla} \times Y_{lm}^{(1)} (kr) \times \vec{X}_{lm} \right] \\
\vec{E} &= \frac{\mu_0}{\varepsilon_0} \sum_{l,m} \left[ \frac{i}{k} \alpha_E (l,m) \vec{\nabla} \times Y_{lm}^{(1)} (kr) \times \vec{X}_{lm} + \alpha_m (l,m) Y_{lm}^{(1)} (kr) \times \vec{X}_{lm} \right]
\end{align*}
\tag{9.122}
\]

where \( \frac{\mu_0}{\varepsilon_0} = " \text{impedance of free space}" \).

The coefficients are

\[
\begin{align*}
\alpha_E (l,m) &= \frac{k^2}{\sqrt{l(l+1)}} \int \nabla_{lm} \left\{ C_{pcr} \left[ j_l (kr) \right] + i k x \cdot \vec{J}_l (kr) \right\} d^3 x \\
\alpha_m (l,m) &= \frac{k^2}{\sqrt{l(l+1)}} \int \nabla_{lm} j_l (kr) \nabla \cdot (x x \vec{J}_l) d^3 x \tag{9.167} \tag{9.168}
\end{align*}
\]

(setting the intrinsic magnetization \( M = 0 \)).
\((a_E, a_M)\) are the (electric, magnetic) multipole coefficients.

Given the coefficients \((a_E, a_M)\), determined by the distributions of \(p\) and \(j\) in the source, one can calculate the total fields \(E, B\) from (9.122). These include the near fields (outside the source), the intermediate fields, and the far (wave) fields. The source is not assumed to be small (good for any \(kr\)).

The electric multipole \((a_E)\) terms have \(E \perp \hat{r}\), but \(E_r \neq 0\).

\[ a_E (1, m), \ m=0, \pm 1 \quad \text{give electric dipole fields} \]
\[ a_E (2, m), \ m=0, \pm 1, \pm 2 \quad \text{give } \text{"quadrupole"} \]

The magnetic multipole \((a_B)\) terms have \(B \perp \hat{r}\), but \(B_r \neq 0\).

\[ a_B (1, m), \ m=0, \pm 1 \quad \text{give magnetic dipole fields} \]
\[ a_B (2, m), \ m=0, \pm 1, \pm 2 \quad \text{"quadrupole"} \]

In the radiation zone \((kr \gg 1)\), the radial function can be approximated

\[ h_{l}^{(1)} \approx (-1)^{l+1} \frac{e^{i k r}}{k r} \quad (9.89) \]

and (9.122) becomes
\[ H = \frac{e^{ikr-\omega t}}{kr} \sum_{l,m} (-i)^{l+1} \left[ a_{l,m} Y_{l,m} + a_{m,l} Y_{m,l} \right] \]

\[ \Xi = 2 \text{Re} H \hat{n} \]

(9.149)

and the period-averaged power radiated per unit solid angle is

\[ \frac{dP}{d\Omega} = \frac{\Xi_0}{2k^2} \left| \sum_{l,m} (-i)^{l+1} \left[ a_{l,m} Y_{l,m} + a_{m,l} Y_{m,l} \right] \right|^2 \]

for a pure multipole of order \((l,m)\), this \(\rightarrow\)

\[ \frac{dP(l,m)}{d\Omega} = \frac{\Xi_0}{2k^2} |a(l,m)|^2 |Y_{l,m}|^2 \]

(9.151)

The angular distributions \(|Y_{l,m}|^2\) are listed in Table 9.1 and plotted in Fig. 9.5 for \((l,m) = (1,0)\) and \((1,\pm 1)\) and for \((l,m) = (2,0), (2,\pm 1), \text{ and } (2,\pm 2)\).

Note that if \(a(l,m)\) is independent of \(m\) for a collection of \(l\)-order multipoles, the angular distribution (9.151) summed over \(m\) is isotropic (independent of \(\theta\) and \(\phi\)), and its integral over solid angle is just

\[ P(l,m) = \frac{\Xi_0}{2k^2} |a(l,m)|^2 \]

(9.154)

(This is generally true for atomic and nuclear radiative transitions in a large sample.)
Figure 9.5  Dipole and quadrupole radiation patterns for pure \((l, m)\) multipoles.
For a collection of different-order multipoles, the total radiated power is

\[ P = \frac{Z_0}{2k^2} \sum_{l,m} \left[ \left| a_E(l,m) \right|^2 + \left| a_m(l,m) \right|^2 \right] \]  (9.155)

For a small source \((ka << 1)\), we can use the first term of a power-series expansion for \(j_0(kr)\):

\[ j_0(kr) \approx \frac{x^l}{(2l+1)!!} \]  (9.88)

where \((2l+1)!! = (2l+1)(2l-1) \cdots (1)\).

Putting this in (9.167), with second term neglected compared to first, and in (9.168), gives

\[ a_E(l,m) = \frac{ckl^2}{i(2l+1)!!} \left( \frac{l+1}{l} \right)^{1/2} Q_{lm} \]  (9.169)

where \(Q_{lm} = \int r^l Y^*_{lm} \rho \, d^3x\)  (9.170)

\[ = \text{electric multipole moment} \]

and

\[ a_m(l,m) = \frac{ckl^2}{(2l+1)!!} \left( \frac{l+1}{l} \right)^{1/2} M_{lm} \]  (9.171)

where \(M_{lm} = \frac{1}{l+1} \int r^l Y^*_{lm} \nabla \cdot (r \times \mathbf{E}) \, d^3x\)  (9.172)

\[ = \text{magnetic multipole moment} \]