Summary of results for electric dipole radiation:
We assumed $\rho(\underline{x},t) = \rho(\underline{x})\,e^{-i\omega t}$, $\underline{J}(\underline{x},t) = \underline{J}(\underline{x})\,e^{-i\omega t}$, and found a general expression for $\underline{A}(\underline{x},t)$:

$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi r}\,e^{i(kr-\omega t)}\int d^3x'\,\underline{J}(\underline{x}')\,e^{-ik\hat{n}\cdot\underline{x}'} \qquad (\times 9.5)$$

Then we assumed the source was small ($ka \ll 1$) and expanded the exponent in a power series, keeping only the $n=0$ term ($e^{-ik\hat{n}\cdot\underline{x}'} \to 1$) to get the radiation fields for electric dipole radiation:

$$\underline{B}_0 = \frac{\mu_0\omega k}{4\pi r}\,e^{i(kr-\omega t)}\,\hat{n}\times\underline{p} \qquad (\approx 9.19)$$

$$\underline{E}_0 = \frac{k^2}{4\pi\epsilon_0 r}\,e^{i(kr-\omega t)}\,\underline{p}_\perp \qquad (\times 9.10)$$

where $\underline{p} \equiv \int \rho(\underline{x}')\,\underline{x}'\,d^3x'$, $\underline{p}_\perp \equiv \hat{n}\times(\underline{p}\times\hat{n})$

One example of this is a center-fed linear antenna (see Jackson p. 412). Another example is a non-relativistic point charge in periodic motion. Assume a point charge q oscillates sinusoidally along z:

$$\rho(\underline{x},t) = q\,\delta(x)\,\delta(y)\,\delta(z - z_0\cos\omega t)$$

$$\rightarrow \underline{p} = q\int d^3x'\,\delta(x)\,\delta(y)\,\delta(z - z_0\cos\omega t)\,\underline{x}'$$

$$= q\,\hat{z}\,z_0\cos\omega t = \hat{z}\,\mathrm{Re}\left[q z_0 e^{-i\omega t}\right]$$

From (9.23), this $\underline{p}$ gives a radiated energy flux

$$\frac{dP}{d\Omega} = \frac{k^4 c}{32\pi^2\epsilon_0} q^2 z_0^2 \sin^2\theta \qquad (x9.11)$$

and, from (9.24), a total power $P = \frac{k^4 c}{12\pi\epsilon_0} q^2 z_0^2$. (x9.12)

So an oscillating charge gives electric dipole radiation. The dipole term dominates higher-order multipoles if the source is small, i.e. $kz_0 \ll 1$ or $\frac{\omega z_0}{c} \ll 1$. But $\omega z_0 = v_0$, the maximum velocity of the oscillating charge. So dipole dominates if $v_0 \ll c$.

This is the essential physics of Thompson scattering.

A point charge moving in a circle is equivalent to two point charges oscillating in $\perp$ directions, 90° out of phase. Indeed, if we double (x9.12) and set $kc = \omega$, $\dot{v} = \omega^2 z_0$, we get the <u>Larmor formula</u> for cyclotron radiation:

$$P = \frac{q^2}{6\pi\epsilon_0 c^3} |\dot{v}|^2$$

(same as Jackson (14.22) in SI units.)

Magnetic Dipole and Electric Quadrupole Radiation

Return to the multipole expansion of (x9.5):

$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi r} e^{i(kr-\omega t)} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x' \, \underline{J}(\underline{x}')(-ik\hat{n}\cdot\underline{x}')^n \qquad (x9.7)$$

We've been looking at the $n=0$ term (electric dipole). This term is usually dominant, but for some cases it is $\equiv 0$ by symmetry, and we have to look at the $n=1$ term:

$$\underline{A}_1(\underline{x},t) = \frac{-ik\mu_0}{4\pi r} e^{i(kr-\omega t)} \int d^3x' \, \underline{J}(\underline{x}') \, \hat{n}\cdot\underline{x}' \qquad (x9.13)$$

The integrand can be split into 2 terms:

$$(\hat{n}\cdot\underline{x}')\underline{J} = \frac{1}{2}\left[(\hat{n}\cdot\underline{x}')\underline{J} + (\hat{n}\cdot\underline{J})\underline{x}'\right] + \frac{1}{2}\left[(\hat{n}\cdot\underline{x}')\underline{J} - (\hat{n}\cdot\underline{J})\underline{x}'\right]$$

The first term is symmetric under interchange of $\underline{x}'$ and $\underline{J}$. The second is antisymmetric. The antisymmetric part is related to

$$\hat{n}\times(\underline{x}'\times\underline{J}) \equiv \underline{x}'(\underline{J}\cdot\hat{n}) - \underline{J}(\underline{x}'\cdot\hat{n})$$

so $$(\hat{n}\cdot\underline{x}')\underline{J} = \frac{1}{2}\left[(\hat{n}\cdot\underline{x}')\underline{J} + (\hat{n}\cdot\underline{J})\underline{x}'\right] + \frac{1}{2}(x'\times\underline{J})\times\hat{n} \qquad (9.31)$$

The vector potential (x9.13) splits into 2 terms:

$$\underline{A}(\underline{x},t) = \underline{A}_{E2} + \underline{A}_{M1} \qquad (x9.14)$$

Where the <u>electric quadrupole</u> term is

$$\underline{A}_{E2} = \frac{-ik\mu_0}{8\pi r} e^{i(kr-\omega t)} \int d^3x' \left[(\hat{n}\cdot\underline{x}')\underline{J} + (\hat{n}\cdot\underline{J})\underline{x}' \right] \quad (x9.15)$$

and the <u>magnetic dipole</u> term is

$$\underline{A}_{M1} = \frac{ik\mu_0}{4\pi r} \hat{n} \times \underline{m} \, e^{i(kr-\omega t)} \quad (x9.16)$$

$$\underline{m} \equiv \frac{1}{2} \int \underline{x}' \times \underline{J}(\underline{x}') d^3x' = \text{magn. dipole moment} \quad (9.34)$$

Same as magnetic dipole moment in magnetostatics (except here it is the part of $\underline{m}$ that oscillates at frequency ω).

For the magnetic dipole term (x9.16),

$$\underline{B}_{M1} = \nabla \times \underline{A}_{M1} = \frac{ik\mu_0}{4\pi} \nabla \times \left[\frac{e^{i(kr-\omega t)}}{r} \hat{n} \times \underline{m} \right]$$

$$= \frac{ik\mu_0}{4\pi} \nabla \left[\frac{e^{i(kr-\omega t)}}{r} \right] \times (\hat{n} \times \underline{m})$$

$$\boxed{\underline{B}_{M1} \simeq \frac{k^2\mu_0}{4\pi r} e^{i(kr-\omega t)} \underline{m}_\perp} \quad (\simeq 9.35)$$

= Magn. field of magn. dipole radiation. $\left(\underline{m}_\perp \equiv (\hat{n} \times \underline{m}) \times \hat{n} \right)$

To get $\underline{E}_{M1}$ from $\underline{B}_{M1}$ use Ampere's law

$$\nabla \times \underline{B} = \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} \rightarrow ik\hat{n} \times \underline{B}_{M1} = -i\omega \mu_0 \epsilon_0 \underline{E}_{M1}$$

$$\underline{E}_{M1} = \frac{-k}{\mu_0 \epsilon_0 \omega} \hat{n} \times \underline{B}_{M1} \qquad (x9.17)$$

Using (9.35) and noting that $\hat{n} \times [\hat{n} \times (\hat{n} \times \underline{m})] = -\hat{n} \times \underline{m}$,

$$\boxed{\underline{E}_{M1} = \frac{-ck^2 \mu_0}{4\pi r} e^{i(kr-\omega t)} \hat{n} \times \underline{m}} \qquad (\approx 9.36)$$

= Electric field of magnetic dipole radiation.

Comparing (9.35, 36) with (9.19, x9.10), note that the radiation fields for electric and magnetic dipole sources are essentially the same, with the following transformations

electric dipole		magnetic dipole
$\underline{p}$	$\longleftrightarrow$	$\underline{m}/c$
$c\underline{B}_{E1}$	$\longleftrightarrow$	$-E_{M1}$
$\underline{E}_{E1}$	$\longleftrightarrow$	$c\underline{B}_{M1}$

The field-line diagram shown last time for $\underline{E}_{E1}$ also applies for $\underline{B}_{M1}$ (in the plane of $\hat{n}$ and the dipole). The radiation patterns are the same ($\frac{dP}{d\Omega} \propto \sin^2\theta$) but the magnitudes of the energy fluxes are quite different:

$$\frac{S_{M1}}{S_{E1}} \sim \frac{|m|^2}{|p|^2 c^2} \sim \left| \frac{\frac{1}{2}\int \underline{x}' \times \underline{J}\, d^3x'}{c\int \rho \underline{x}'\, d^3x'} \right|^2 \sim \frac{|\underline{J}|^2}{|\rho|^2 c^2}$$

charge conservation $\Rightarrow$ $\nabla \cdot \underline{J} = i\omega\rho$

If $\nabla \cdot \underline{J} \sim J/a$ then $\dfrac{|J|}{|\rho|} \sim \omega a$ and

$$\frac{S_{M1}}{S_{E1}} \sim \frac{\omega^2 a^2}{c^2} = k^2 a^2 \ll 1 \text{ for small source.}$$

$\therefore$ Electric dipole radiation dominates <u>unless</u> $\underline{J}$ has no divergence ($\rho = 0$). If $\rho \neq 0$,

$$\frac{J}{\rho} \sim v \qquad \frac{S_{M1}}{S_{E1}} \sim \frac{v^2}{c^2} \ll 1 \text{ for non-relativistic motion.}$$

<u>Electric Quadrupole Radiation</u>

$$(\text{x}9.15) \rightarrow \underline{A}_{E2} = \frac{-ik\mu_0}{8\pi r} e^{i(kr-\omega t)} \; \hat{n} \cdot \int (\underline{x}'\underline{J} + \underline{J}\underline{x}') d^3x' \qquad (\text{x}9.18)$$

To evaluate this integral, first consider the quantity

$$I_{ij} \equiv \int d^3x' \, \rho(\underline{x}') \, x_i' x_j'$$

Using charge conservation $\dfrac{\partial J_k}{\partial x_k} = i\omega\rho$,

$$I_{ij} = \frac{1}{i\omega} \int \frac{\partial J_k}{\partial x_k'} x_i' x_j' \, d^3x'$$

$$= \frac{1}{i\omega} \int \left[\frac{\partial}{\partial x_k'} (x_i' x_j' J_k) - \delta_{ik} x_j' J_k - \delta_{jk} x_i' J_k \right] d^3x'$$

The first term integrates to zero because $J = 0$ outside the source. Thus

$$\int [x_j' J_i + x_i' J_j] d^3x' = -i\omega \int \rho(\underline{x}') x_i' x_j' d^3x'$$

and substituting this in (x9.18) gives

$$\underline{A}_{E2}(\underline{x},t) = \frac{-\omega k \mu_0}{8\pi r} e^{i(kr-\omega t)} \hat{n} \cdot \int \underline{x}'\underline{x}' \rho(\underline{x}') d^3x'$$

The radiation magn. field is then

$$\underline{B}_{E2}(\underline{x},t) = \nabla \times \underline{A}_{E2} = ik\hat{n} \times \underline{A}_{E2}$$

$$= \frac{-i\omega k^2 \mu_0}{8\pi r} e^{i(kr-\omega t)} \hat{n} \times \left\{ \hat{n} \cdot \int \underline{x}'\underline{x}' \rho(\underline{x}') d^3x' \right\}$$

$$= \frac{-i\omega k^2 \mu_0}{24\pi r} e^{i(kr-\omega t)} \hat{n} \times \left\{ \hat{n} \cdot \int (3\underline{x}'\underline{x}' - \underline{\underline{1}}\, x'^2) \rho(\underline{x}') d^3x' \right\}$$

using the fact that $\hat{n} \times (\hat{n} \cdot \underline{\underline{1}}) = \hat{n} \times \hat{n} = 0$.

Recalling the definition of the electric quadrupole moment (from L 3)

$$\underline{\underline{Q}} = \int [3\underline{x}'\underline{x}' - \underline{\underline{1}}\, x'^2] \rho(\underline{x}') d^3x' \qquad (4.9 = 9.41)$$

we can write

$$\boxed{\underline{B}_{E2}(\underline{x},t) = \frac{-ick^3 \mu_0}{24\pi r} e^{i(kr-\omega t)} \hat{n} \times (\hat{n} \cdot \underline{\underline{Q}})} \qquad (\approx 9.44)$$

= Magn. field of electric quadrupole radiation.

To get $\underline{E}_{E2}$ from $\underline{B}_{E2}$ use Ampere again

$$\underline{E}_{E2} = \frac{-k}{\omega\mu_0\epsilon_0}\,\hat{n}\times\underline{B}_{E2}$$

$$= \frac{ik^3}{24\pi\epsilon_0 r}\,e^{i(kr-\omega t)}\,\hat{n}\times\left[\hat{n}\times(\hat{n}\cdot\underline{\underline{Q}})\right]$$

$$\boxed{\underline{E}_{E2} = \frac{-ik^3}{24\pi\epsilon_0 r}\,e^{i(kr-\omega t)}\left[\hat{n}\cdot\underline{\underline{Q}}\right]_\perp} \qquad (x9.19)$$

= elect. field of elect. quadrupole radiation.