Radiating Systems (Jackson 9.1 - 9.2)

Consider radiation from a sinusoidally oscillating localized source:

$$\begin{align*}
\rho(x,t) &= \rho(x) e^{-i\omega t} \\
\mathbf{J}(x,t) &= \mathbf{J}(x) e^{-i\omega t}
\end{align*} \quad (9.1)$$

$\rho(x,t)$ and $\mathbf{J}(x,t)$ are not independent but are related by the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = -i\omega \rho \quad (x9.1)$$

It is convenient to work with the potentials $\Phi$ and $A$ instead of the fields $E$ and $B$, and use the Lorentz gauge condition $\nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \quad (6.14)$

Recall that $\Phi, A$ satisfy inhomogeneous wave equations

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\rho / \varepsilon_0 \quad (6.15)$$

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A = -\mu_0 \mathbf{J} \quad (6.16)$$

with general solutions

$$\Phi(x,t) = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\Phi(x',t) - \frac{|x-x'|}{c}}{|x-x'|} \quad (6.98)$$

$$A(x,t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{J(x',t) - \frac{|x-x'|}{c}}{|x-x'|} \quad (6.98)$$
Substituting (9.1) in (6.48) gives

\[ \Phi(x, t) = \frac{e^{i\omega t}}{4\pi \varepsilon_0} \int d^3 x' \frac{\rho(x') e^{ik|x-x'|}}{|x-x'|} \] (x.9.2)

\[ A(x, t) = \frac{\mu_0 e^{-i\omega t}}{4\pi} \int d^3 x' \frac{J(x') e^{ik|x-x'|}}{|x-x'|} \]

where \( k = \frac{\omega}{c} \).

Let's look at the radiation field, where \(|x| \gg |x'|\):

\[ |x-x'| = \sqrt{|x|^2 - 2x \cdot x' + |x'|^2} = r \sqrt{1 - \frac{2\hat{n} \cdot x'}{r} + \frac{|x'|^2}{r^2}} \]

where \( r = |x| \), \( \hat{n} = \frac{x}{r} \).

\[ |x-x'| \approx r \left[ 1 + \frac{1}{8} \left( -2 \frac{\hat{n} \cdot x'}{r} + \frac{|x'|^2}{r^2} \right) - \frac{1}{8} \left( -2 \frac{\hat{n} \cdot x'}{r} + \frac{|x'|^2}{r^2} \right) \right] \]

\[ \approx r - \hat{n} \cdot x' + o\left(\frac{|x'|^2}{r}\right) \] (x.9.4)

Neglect terms of order \( \frac{k|x'|^2}{r} \) (and higher) in the exponents of (x.9.2). Also set \(|x-x'| \approx r\) in the denominators. (The neglected terms vary faster than \(1/r\) and are therefore, by definition, not part of the radiation field potentials.) This gives

\[ \Phi(x, t) \approx \frac{e^{i(kr-\omega t)}}{4\pi \varepsilon_0 r} \int d^3 x' \rho(x') e^{i2k \hat{n} \cdot x'} \] (x.9.5)

\[ A(x, t) \approx \frac{\mu_0 e^{-i(kr-\omega t)}}{4\pi r} \int d^3 x' J(x') e^{-i2k \hat{n} \cdot x'} \]
Equations (x9.5) give the radiation field of any sinusoidally oscillating source. (Size of source can be \( \gg \) wavelength \( 2\pi/k \), so far.)

There are 3 length scales:

\[ |x'| \sim a = \text{source size} \]

\[ r = \text{distance from source to observer} \]

\[ 1/k = \text{wavelength }/2\pi \]

We've already assumed \( r \gg a \). Suppose also

\[ k a \ll 1 \]  \hspace{1cm} (x9.6) \]

i.e., source size \( \ll \) wavelength. Then the exponential in (x9.5) can be expressed as a rapidly converging power series:

\[ \Phi(x, t) = \frac{e^{i(kr-\omega t)}}{4\pi r} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \frac{d^3x'}{d^3x} \rho(x')( -i k\mathbf{n} \cdot x') \right)^n \]  \hspace{1cm} (x9.7) \]

\[ A(x, t) = \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \frac{d^3x'}{d^3x} \mathcal{J}(x')( -i k\mathbf{n} \cdot x') \right)^n \]

Consider the \( n=0 \) terms:

\[ \Phi_0(x, t) = \frac{e^{i(kr-\omega t)}}{4\pi r} \int d^3x' \rho(x') \]  \hspace{1cm} (x9.8) \]

\[ A_0(x, t) = \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r} \int d^3x' \mathcal{J}(x') \]

But, \( \int d^3x' \rho(x') = \text{total charge} = \text{constant} \) for a localized source. (by definition), so the \( n=0 \) term for \( \Phi \) is zero unless \( \omega = 0 \) (\( \Rightarrow k = 0 \)), i.e., static monopole field (not radiation field).
There is no monopole radiation field.

What about \( A_0(x,t) \)? The integral \( \int d^3x' J(x') = 0 \) in electrostatics, but it can be \( \neq 0 \) in the presence of a time-varying charge density. In fact we can write \( \int J d^3x' \) in terms of \( p \). Consider

\[
\int d^3x' \frac{\partial}{\partial x_j} (x' \cdot J_j(x')) = \int d^3x' \delta_{ij} J_j(x') + \int d^3x' x'_i \frac{\partial J_j}{\partial x_j} = 0 \tag{9.14}
\]

by integration by parts, since \( J = 0 \) at boundary of source.

\[
\therefore \int d^3x' J(x') = -\int d^3x' x' \cdot \nabla J \tag{9.14}
\]

But charge conservation (9.1) \( \Rightarrow x' \cdot J = i\omega p \)

\[
\therefore \int d^3x' J(x') = -i\omega \int d^3x' \rho(x') x' = -i\omega p \tag{9.9}
\]

Where \( p = \int d^3x' x' \rho(x') = \text{electric dipole moment} \tag{9.17} \)

(same as in electrostatics).

Substituting (9.9) in (9.8) gives

\[
A_0(x,t) = -\frac{i\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \tag{9.16}
\]
(9.16) describes electric dipole radiation, the most common form in nature. Examples include linear antennas, atomic transitions, Thompson scattering, and cyclotron radiation.

Notes:

- Jackson, after eqn. (9.3), says "a sinusoidal time dependence is understood." This means there is an invisible factor $e^{-i\omega t}$ on the right side of (9.3), (9.6), (9.8), (9.9), (9.11), (9.13), (9.16), (9.18), (9.19), (9.20), and subsequent equations that have $E$, $A$, $E_z$, or $H$ on the left side. But not (9.17). The electric dipole moment defined here is the static one given by $p(x)$. The actual dipole moment of the source is of course time-dependent ($e^{-i\omega t}$) because of the time-dependence of $p(x,t) = p(x)e^{-i\omega t}$ assumed in (9.1).

- Jackson has a factor $\rho$ missing from the denominator on the right side of the second un-numbered eqn. on top of p. 916.

We can calculate the radiation fields $E$ and $B$ from (9.16). (Don't need the corresponding expression for $\Phi_0(x,t)$. ) Jackson uses $H = \frac{1}{\mu_0} B$. It's easier to start with $B$ then derive $E$ from it.
\[ B_o = \nabla \times A_o = q \times \left( -\frac{i \mu_0 \omega}{4\pi} \mathbf{p} \frac{e^{i(kr-\omega t)}}{r} \right) \]
\[ = -\frac{i \mu_0 \omega}{4\pi} e^{-i\omega t} \nabla \left( \frac{e^{ikr}}{r} \right) \times \mathbf{p} \]

(since \( \mathbf{p} \) is constant vector)

\[ \nabla \left( \frac{e^{ikr}}{r} \right) = i k \mathbf{\hat{n}} \frac{e^{ikr}}{r} - \mathbf{\hat{n}} \frac{e^{ikr}}{r^2} \]

\[ = k \mathbf{\hat{n}} \frac{e^{ikr}}{r} \left( i - \frac{1}{kr} \right) \approx i k \mathbf{\hat{n}} \frac{e^{ikr}}{r} \]

\[ \Rightarrow B_o \approx \frac{\mu_0 \omega k}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \mathbf{\hat{n}} \times \mathbf{p} \]

\[ \approx \frac{\mu_0 \omega k}{4\pi} \mathbf{\hat{n}} \times \mathbf{p} \quad (kr \gg 1) \]

\[ \approx \frac{\mu_0 \omega k}{4\pi} \mathbf{\hat{n}} \times \mathbf{p} = \text{Radiation magnetic field of electric dipole.} \]

We can get \( E_o \) from \( B_o \) using Faraday's law:

\[ \nabla \times B_o = \mu_0 \epsilon_0 \frac{\partial E_o}{\partial t} = -i \omega \mu_0 \epsilon_0 E_o \]

from (9.19), \( \nabla \times B_o = \frac{\mu_0 \omega k}{4\pi} \nabla \left[ \frac{e^{ikr}}{r} \mathbf{\hat{n}} \times \mathbf{p} \right] e^{-i\omega t} \]

\[ = \frac{\mu_0 \omega k}{4\pi} \mathbf{\hat{n}} \cdot \nabla \frac{e^{ikr}}{r} - \mathbf{\hat{n}} \cdot \nabla \frac{e^{ikr}}{r} \]

(since \( \mathbf{p} \) is constant and \( \mathbf{p} \cdot \nabla \mathbf{\hat{n}} = 0 \))

\[ \nabla \left( \frac{e^{ikr}}{r} \right) \approx i k \mathbf{\hat{n}} \frac{e^{ikr}}{r} \quad \text{(above)} \Rightarrow \]

\[ \nabla \times B_o \approx \frac{i \mu_0 \omega k^2}{4\pi r} \left[ \mathbf{\hat{n}} (p \cdot \mathbf{\hat{n}}) - p \right] \]

\[ \approx -p_{\perp} = -\text{comp} \cdot p \perp \mathbf{\hat{n}}. \]
\[ E_0 = \frac{k^2}{4\pi\varepsilon_0 r} e^{i(kr-wt)} p \]

= radiation electric field of an electric dipole.

Things to note:
- \( E_0 \) and \( B_0 \) are both \( \perp \hat{n} \). (transverse wave)
- \( E_0 \times B_0 \parallel p \times (\hat{n} \times p) = p^2 \hat{n} \) = radially outward.
- \( p \to 0 \) for \( \hat{n} \parallel p \Rightarrow \) no radiation \( \parallel p \).
- \( E_0 \) in the plane of \( p \) and \( \hat{n} \).
- \( B_0 \) is \( \perp \) to \( \hat{n} \).

Fig. 14-5. Electric field lines produced by an oscillating dipole.

From Panovsky & Phillips
The time-average Poynting energy flux (averaged over a wave period) per unit solid angle $\Omega$ is

$$\frac{dP}{d\Omega} = r^2 \langle \hat{n} \cdot \frac{\mathbf{E}_0 \times \mathbf{B}_0}{\mu_0} \rangle - \frac{r^2}{2\mu_0} \hat{n} \cdot \text{Re}(\mathbf{E}_0^* \times \mathbf{B}_0^*)$$

$$= \frac{r^2}{2\mu_0} \hat{n} \cdot \text{Re} \left[ \frac{k^2}{4\pi\varepsilon_0 \nu} e^{i(kr-\omega t)} \mathbf{p}_\perp^* \times \frac{\mathbf{p}_\perp}{4\pi r} e^{i \frac{\nu}{4}} (\hat{n} \times \mathbf{p}_\perp) \right]$$

$$= \frac{\omega k^3}{32\pi^2 \varepsilon_0} \text{Re} \left\{ \hat{n} \cdot \left[ \mathbf{p}_\perp^* \times (\hat{n} \times \mathbf{p}_\perp) \right] \right\} \quad (9.41)$$

$$= \frac{ck^4}{32\pi^2 \varepsilon_0} |\mathbf{p}_\perp|^2 \quad (9.22)$$

If $\mathbf{p}^*$ is $\parallel \mathbf{p}$ (all components of $\mathbf{p}$ have the same phase), then $|\mathbf{p}_\perp|^2 = \mathbf{p}^2 \sin^2 \theta$ where $\theta$ = angle from $\mathbf{p}$. Then

$$\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2 \varepsilon_0} \mathbf{p}^2 \sin^2 \theta \quad (9.23)$$

We can integrate this over solid angle ($d\Omega = 2\pi \sin \theta \, d\theta$) to get the total radiated power

$$P = \frac{ck^4}{32\pi^2 \varepsilon_0} \mathbf{p}^2 2\pi \int_0^\pi \sin^3 \theta \, d\theta$$

$$= \frac{ck^4 \mathbf{p}^2}{16\pi \varepsilon_0} \left[ -\frac{1}{3} \cos \theta \right]_0^\pi$$

$$= \frac{ck^4 \mathbf{p}^2}{12\pi \varepsilon_0} \quad (9.24)$$