PHYS 532 L22

<u>Radiating Systems</u> (Jackson 9.1 - 9.2)

Consider radiation from a sinusoidally oscillating localized source:

$$\left.\begin{aligned} \rho(\underline{x},t) &= \rho(\underline{x})\, e^{-i\omega t} \\ \underline{J}(\underline{x},t) &= \underline{J}(\underline{x})\, e^{-i\omega t} \end{aligned}\right\} \quad (9.1)$$

$\rho(\underline{x},t)$ and $\underline{J}(\underline{x},t)$ are not independent but are related by the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\underline{J} = 0 \quad\Rightarrow\quad \nabla\cdot\underline{J} = i\omega\rho \qquad (\times 9.1)$$

It is convenient to work with the potentials Φ and $\underline{A}$, instead of the fields $\underline{E}$ and $\underline{B}$, and use the Lorenz gauge condition $\nabla\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$ (6.14)

Recall that Φ, $\underline{A}$ satisfy inhomogeneous wave equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\epsilon_0 \qquad (6.15)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underline{A} = -\mu_0\underline{J} \qquad (6.16)$$

with general solutions

$$\left.\begin{aligned} \Phi(\underline{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \,\frac{\rho\left(\underline{x}', t - \frac{|\underline{x}-\underline{x}'|}{c}\right)}{|\underline{x}-\underline{x}'|} \\ \underline{A}(\underline{x},t) &= \frac{\mu_0}{4\pi}\int d^3x' \,\frac{\underline{J}\left(\underline{x}', t - \frac{|\underline{x}-\underline{x}'|}{c}\right)}{|\underline{x}-\underline{x}'|} \end{aligned}\right\} (6.48)$$

Substituting (9.1) in (6.48) gives

$$\left.\begin{aligned}\underline{\Phi}(\underline{x},t) &= \frac{e^{-i\omega t}}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\underline{x}')e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \\ \underline{A}(\underline{x},t) &= \frac{\mu_0 e^{-i\omega t}}{4\pi}\int d^3x' \, \frac{\underline{J}(\underline{x}')e^{ik|\underline{x}-\underline{x}'|}}{|x-x'|}\end{aligned}\right\} \quad (x\,9.2)$$

where $k = \omega/c$.

Let's look at the <u>radiation field</u>, where $|\underline{x}| \gg |\underline{x}'|$:

$$|\underline{x}-\underline{x}'| = \sqrt{|\underline{x}|^2 - 2\underline{x}\cdot\underline{x}' + |\underline{x}'|^2} = r\sqrt{1 - \frac{2\hat{n}\cdot\underline{x}'}{r} + \frac{|\underline{x}'|^2}{r^2}}$$

where $r \equiv |\underline{x}|$, $\hat{n} \equiv \frac{\underline{x}}{r}$

$$|\underline{x}-\underline{x}'| \approx r\left[1 + \frac{1}{2}\left(-2\frac{\hat{n}\cdot\underline{x}'}{r} + \frac{|\underline{x}'|^2}{r^2}\right) - \frac{1}{8}\left(-2\frac{\hat{n}\cdot\underline{x}'}{r} + \frac{|\underline{x}'|^2}{r^2}\right)^2 \cdots\right]$$

$$\approx r - \hat{n}\cdot\underline{x}' + O\left(\frac{|x'|^2}{r}\right) \quad (x\,9.4)$$

Neglect terms of order $\frac{k|x'|^2}{r}$ (and higher) in the exponents of $(x\,9.2)$. Also set $|\underline{x}-\underline{x}'| \approx r$ in the denominators. (The neglected terms vary faster than $1/r$ and are therefore, by definition, not part of the radiation field potentials.) This gives

$$\left.\begin{aligned}\underline{\Phi}(\underline{x},t) &\approx \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0 r}\int d^3x' \, \rho(\underline{x}')e^{-ik\hat{n}\cdot\underline{x}'} \\ \underline{A}(\underline{x},t) &\approx \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r}\int d^3x' \, \underline{J}(\underline{x}')e^{-ik\hat{n}\cdot\underline{x}'}\end{aligned}\right\} \quad (x\,9.5)$$

Equations (x9.5) give the radiation field of <u>any</u> sinusoidally oscillating source. (Size of source can be >> wavelength $2\pi/k$, so far.)

There are 3 length scales:

$|\underline{x}'| \sim a$ = source size

r = distance from source to observer

$1/k$ = wavelength $/2\pi$

We've already assumed $r \gg a$. Suppose also

$$ka \ll 1 \qquad (x9.6)$$

i.e., source size << wavelength. Then the exponential in (x9.5) can be expressed as a rapidly converging power series:

$$\Phi(\underline{x},t) = \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x'\,\rho(\underline{x}')(-ik\hat{n}\cdot\underline{x}')^n$$

$$\underline{A}(\underline{x},t) = \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r}\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x'\,\underline{J}(\underline{x}')(-ik\hat{n}\cdot\underline{x}')^n \qquad (x9.7)$$

Consider the $n=0$ terms:

$$\Phi_0(\underline{x},t) = \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0 r}\int d^3x'\,\rho(\underline{x}')$$

$$\underline{A}_0(\underline{x},t) = \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r}\int d^3x'\,\underline{J}(\underline{x}') \qquad (x9.8)$$

But $\int d^3x'\,\rho(\underline{x}')$ = total charge = constant for a localized source (by definition), so the $n=0$ term for Φ is zero unless $\omega=0$ ($\Rightarrow k=0$), i.e., static monopole field (not radiation field).

<u>There is no monopole radiation field.</u>

What about $\underline{A}_0(\underline{x},t)$? The integral $\int d^3x'\, \underline{J}(\underline{x}') = 0$ in electrostatics, but it can be $\neq 0$ in the presence of a time-varying charge density. In fact we can write $\int \underline{J}\, d^3x'$ in terms of ρ. Consider

$$\underbrace{\int d^3x' \frac{\partial}{\partial x_j'}\left(x_i' J_j(\underline{x}')\right)}_{=0} = \int d^3x' \delta_{ij} J_j(\underline{x}') + \int d^3x' x_i' \frac{\partial J_j}{\partial x_j'}$$

$= 0$ by integration by parts, since $\underline{J} = 0$ at boundary of source.

$$\therefore \int d^3x'\, \underline{J}(\underline{x}') = -\int d^3x'\, \underline{x}'\, \nabla'\cdot\underline{J} \qquad (\sim 9.14)$$

But charge conservation (x9.1) $\Rightarrow$ $\nabla'\cdot\underline{J} = i\omega\rho$

$$\therefore \int d^3x'\, \underline{J}(\underline{x}') = -i\omega \int d^3x'\, \rho(\underline{x}')\, \underline{x}' \equiv -i\omega\, \underline{p} \qquad (x9.9)$$

Where $\underline{p} \equiv \int d^3x'\, \underline{x}' \rho(\underline{x}')$ = electric dipole moment (9.17)

(same as in electrostatics).

Substituting (x9.9) in (x9.8) gives

$$\underline{A}_0(\underline{x},t) = \frac{-i\mu_0\omega}{4\pi}\, \underline{p}\, \frac{e^{i(kr-\omega t)}}{r} \qquad (9.16)$$

(9.16) describes <u>electric dipole radiation</u>, the most common form in nature. Examples include linear antennas, atomic transitions, Thompson scattering, and cyclotron radiation.

<u>Notes</u>:

- Jackson, after eqn. (9.3), says "a sinusoidal time dependence is understood." This means there is an invisible factor $e^{-i\omega t}$ on the right side of (9.3), (9.6), (9.8), (9.9), (9.11), (9.13), (9.16), (9.18), (9.19), (9.20), and subsequent equations that have $\underline{\Phi}$, $\underline{A}$, $\underline{E}$, or $\underline{H}$ on the left side. But not (9.17). The electric dipole moment defined here is the static one given by $\rho(\underline{x}')$. The actual dipole moment of the source is of course time-dependent ($e^{-i\omega t}$) because of the time-dependence of $\rho(\underline{x}', t) = \rho(\underline{x}')e^{-i\omega t}$ assumed in (9.1).

- Jackson has a factor r missing from the denominator on the right side of the second un-numbered eqn. on top of p. 410.

We can calculate the radiation fields $\underline{E}$ and $\underline{B}$ from (9.16). (Don't need the corresponding expression for $\Phi_0(\underline{x}, t)$.) Jackson uses $\underline{H} = \frac{1}{\mu_0}\underline{B}$. It's easier to start with $\underline{B}$ then derive $\underline{E}$ from it.

$$\underline{B}_0 = \nabla \times \underline{A}_0 = \nabla \times \left(\frac{-i\mu_0 \omega}{4\pi} \underline{p} \frac{e^{i(kr-\omega t)}}{r} \right)$$

$$= \frac{-i\mu_0\omega}{4\pi} e^{-i\omega t} \nabla\left(\frac{e^{ikr}}{r}\right) \times \underline{p}$$

(since $\underline{p}$ = constant vector)

$$\nabla\left(\frac{e^{ikr}}{r}\right) = ik\hat{n}\frac{e^{ikr}}{r} - \hat{n}\frac{e^{ikr}}{r^2}$$

$$= k\hat{n}\frac{e^{ikr}}{r}\left(i - \frac{1}{kr}\right) \approx ik\hat{n}\frac{e^{ikr}}{r}$$

$(kr \gg 1)$

$$\rightarrow \boxed{\underline{B}_0 \approx \frac{\mu_0 \omega k}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \hat{n} \times \underline{p}} \qquad (\sim 9.19)$$

= Radiation mag. field of electric dipole.

We can get $\underline{E}_0$ from $\underline{B}_0$ using Faraday's law:

$$\nabla \times \underline{B}_0 = \mu_0 \epsilon_0 \frac{\partial \underline{E}_0}{\partial t} = -i\omega \mu_0 \epsilon_0 \underline{E}_0$$

from (9.19), $\nabla \times \underline{B}_0 = \frac{\mu_0 \omega k}{4\pi} \nabla \times \left[\frac{e^{ikr}}{r} \hat{n} \times \underline{p}\right] e^{-i\omega t}$

$$= \frac{\mu_0 \omega k}{4\pi} \hat{n}\, \underline{p}\cdot\nabla \frac{e^{ikr}}{r} - \underline{p}\, \hat{n}\cdot\nabla \frac{e^{ikr}}{r}$$

(Since $\underline{p}$ = const and $\underline{p}\cdot\nabla\hat{n} = 0$)

$$\nabla\left(\frac{e^{ikr}}{r}\right) \approx ik\hat{n}\frac{e^{ikr}}{r} \qquad \text{(above)} \rightarrow$$

$$\nabla \times \underline{B}_0 \approx \frac{i\mu_0 \omega k^2}{4\pi r} \Big[\underbrace{\hat{n}(\underline{p}\cdot\hat{n}) - \underline{p}}_{\equiv -p_\perp = -\text{comp. } p \perp \hat{n}.} \Big]$$

$$\rightarrow \boxed{\underline{E}_0 = \frac{k^2}{4\pi\epsilon_0 r}\, e^{i(kr-\omega t)}\, \underline{p}_\perp} \qquad (x9.10)$$

= radiation electric field of an electric dipole.

Things to note:

- $\underline{E}_0$ and $\underline{B}_0$ are both $\perp \hat{n}$. (transverse wave)
- $\underline{E}_0 \times \underline{B}_0 \parallel \underline{p}_\perp \times (\hat{n} \times \underline{p}_\perp) = p_\perp^2\, \hat{n}$ = radially outward.
- $\underline{p}_\perp \to 0$ for $\hat{n} \parallel \underline{p}$ $\Rightarrow$ no radiation $\parallel \underline{p}$.
- $\underline{E}_0$ is in the plane of $\underline{p}$ and $\hat{n}$.
- $\underline{B}_0$ is $\perp$ to " " " " .

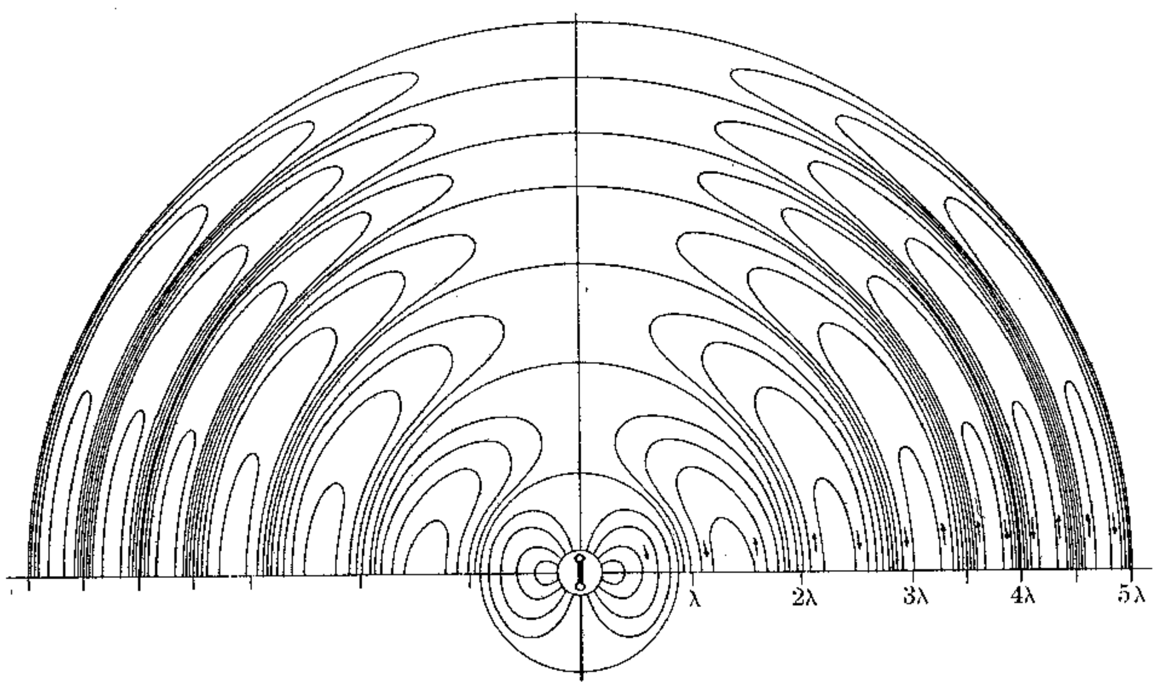

FIG. 14–5. Electric field lines produced by an oscillating dipole.

From Panovsky & Phillips

The time-average Poynting energy flux (averaged over a wave period) per unit solid angle Ω is

$$\frac{dP}{d\Omega} = r^2 \left\langle \hat{n} \cdot \frac{\underline{E}_0 \times \underline{B}_0}{\mu_0} \right\rangle = \frac{r^2}{2\mu_0} \hat{n} \cdot \mathrm{Re}\left(\underline{E}_0^* \times \underline{B}_0\right)$$

$$= \frac{r^2}{2\mu_0} \hat{n} \cdot \mathrm{Re}\left[\frac{k^2}{4\pi\epsilon_0 r} e^{-i(kr-\omega t)} \underline{p}_\perp^* \times \frac{\mu_0 \omega k}{4\pi r} e^{i()} (\hat{n} \times \underline{p}_\perp)\right]$$

$$= \frac{\omega k^3}{32\pi^2 \epsilon_0} \mathrm{Re}\left\{\hat{n} \cdot \left[\underline{p}_\perp^* \times (\hat{n} \times \underline{p}_\perp)\right]\right\}$$

$$= \frac{c k^4}{32\pi^2 \epsilon_0} |p_\perp|^2 \qquad (\times 9.11) = (9.22)$$

If $\underline{p}^*$ is $\parallel \underline{p}$ (all components of $\underline{p}$ have the same phase), then $|p_\perp|^2 = p^2 \sin^2\theta$ where θ = angle from $\underline{p}$. Then $\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2 \epsilon_0} p^2 \sin^2\theta$ (9.23)

We can integrate this over solid angle ($d\Omega = 2\pi \sin\theta \, d\theta$) to get the total radiated power

$$P = \frac{ck^4}{32\pi^2\epsilon_0} p^2 \, 2\pi \int_0^\pi \sin^3\theta \, d\theta$$

$$= \frac{ck^4 p^2}{16\pi\epsilon_0} \left[-\frac{1}{3} \cos\theta \left(2 + \sin^2\theta\right)\right]_0^\pi$$

$$= \frac{ck^4 p^2}{12\pi\epsilon_0} \qquad (9.24)$$