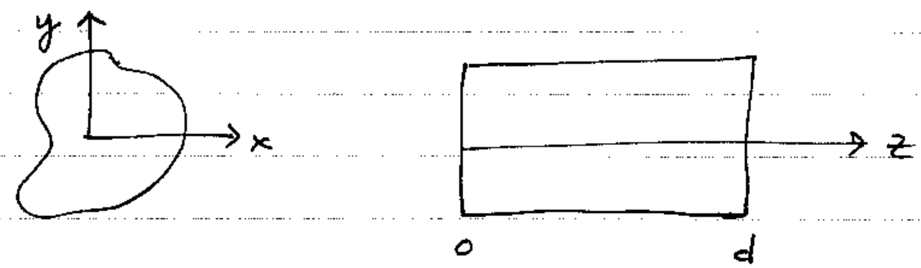
Resonant Cavities

A resonant cavity can be a volume of arbitrary shape enclosed by a conducting surface.
The simplest case to analyze mathematically is a conducting cylinder (arbitrary cross-sectional shape) with conducting end caps ⊥ axis:



i.e., a section of a waveguide with end caps.
Like a waveguide, it also has TM and TE modes, but because of the end caps, the enclosed waves are standing waves rather than travelling waves. Standing waves can exist only at certain discrete frequencies (resonant frequencies), determined by the length d.

The transverse field structure is the same as for a waveguide, except that now we require $\underline{E}_t = 0$ at $z=0$ and at $z=d$.

For $\begin{Bmatrix} TM \\ TE \end{Bmatrix}$ mode, this means $\begin{Bmatrix} \nabla_t E_{z0} \\ \nabla_t B_{z0} \end{Bmatrix} = 0$ at $z=0, d$

This means $\begin{Bmatrix} E_z' \\ B_z \end{Bmatrix} \propto \sin\left(\frac{p\pi z}{d}\right)$ where p = integer,

where $E_z' = \partial E_z / \partial z$.

Jackson works out the case of a right circular cylinder. Let's continue here with a rectangular cylinder (e.g., microwave oven), with dimensions a and b on the $\hat{x}$ and $\hat{y}$ axes. For the open-ended waveguide we had (L19, p. 5)

$$\omega^2 = \frac{1}{\mu\epsilon}\left(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$$

so propagation $(k^2 > 0)$ required

$$\omega^2 > \omega_{mn}^2 \equiv \frac{1}{\mu\epsilon}\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$$

But now, instead of $k^2 > 0$, we have the standing-wave condition $k^2 = \frac{p^2\pi^2}{d^2}$ so instead of a range of frequencies we have a discrete set of resonant frequencies:

$$\omega^2 = \omega_{mnp}^2 \equiv \frac{1}{\mu\epsilon}\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{d^2}\right)$$

for either the TM or TE mode.

For the TM mode the eigenfunctions are given by

$$E_z \propto \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{d}$$

so a nontrivial solution requires $(m, n, p) > 0$.

For the TE mode, the eigenfunctions are given by

$$B_z \propto \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\frac{p\pi z}{d}$$

So, as before, we can have m or n (not both) $= 0$, and the lowest resonant frequency (for $a > b$) is given by

$$\omega_{1,0,1}^2 = \frac{1}{\mu\epsilon}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{d^2}\right) \qquad \text{(TE mode)}.$$

Lowest resonant frequency for a TM mode is given by

$$\omega_{1,1,1}^2 = \frac{1}{\mu\epsilon}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{d^2}\right) \qquad \text{(TM mode)}$$

In general, the lowest resonant frequency, in an order-of-magnitude estimate, is

$$\omega \sim \frac{\pi}{\sqrt{\mu\epsilon}} \frac{1}{l_{min}}$$

where l_{min} = <u>smallest</u> dimension of cavity.

Q of a cavity.

Energy loss to the conducting walls causes the waves to decay with time. The Q (quality) of a resonant cavity at a particular resonant frequency ω_0 is defined by

$$Q = \omega_0 \frac{\text{stored energy}}{\text{Power loss}}$$

so an efficient cavity has $Q \gg 1$.
If U is the wave energy density* and P is the power lost to the walls*, then $Q = \omega_0 U / P$.
But energy conservation implies $P = -dU/dt$.
$\therefore$ U satisfies the d.e. (* per unit length)

$$\left.\begin{aligned} \frac{dU}{dt} &= -\frac{\omega_0}{Q} U \\ \text{with solution} \quad U(t) &= U_0 e^{-\omega_0 t / Q} \end{aligned}\right\} \quad (8.87)$$

Since $U \propto E^2$ this implies

$$E(t) = E_0 e^{-\frac{\omega_0 t}{2Q}} e^{-i(\omega_0 + \Delta\omega)t} \qquad (t > 0)$$

where $\Delta\omega$ is a frequency shift due to non-ideal boundary conditions at the walls.

Taking the Fourier transform of $E(t)$ gives the frequency spectrum

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\left[-\frac{\omega_0}{2Q} + i(\omega - \omega_0 - \Delta\omega)\right]t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\frac{\omega_0}{2Q} - i(\omega - \omega_0 - \Delta\omega)}$$

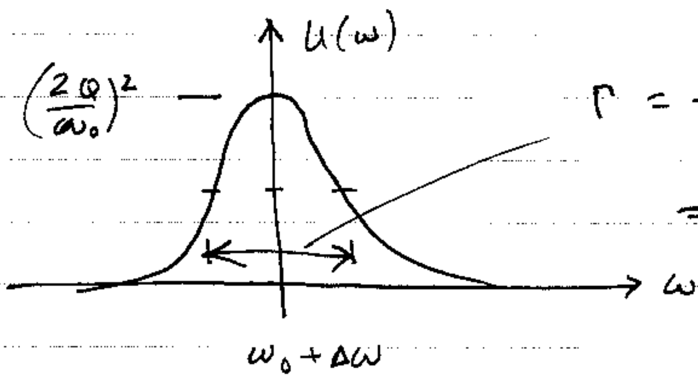
$$= \frac{1}{\sqrt{2\pi}} \frac{\frac{\omega_0}{2Q} + i(\omega - \omega_0 - \Delta\omega)}{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0 - \Delta\omega)^2}$$

so the energy spectrum is

$$U(\omega) \propto E(\omega)^* E(\omega) \propto \frac{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0 - \Delta\omega)^2}{\left[\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0 - \Delta\omega)^2\right]^2}$$

$$\propto \frac{1}{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0 - \Delta\omega)^2} \qquad (8.90)$$

which has a resonance line shape



Γ = full width at half-maximum $= \frac{\omega_0}{Q}$

According to Jackson, "it is easy to show that"

$$Q = \frac{\mu}{\mu_c}\left[\frac{V}{S\delta}\right](\text{geometry factor}) \qquad (8.96)$$

where μ (μ_c) is the permeability of the cavity interior (walls), V = volume, S = surface area, δ = skin depth of conducting walls, and the geometric factor is dimensionless, of order unity, depending only on the geometry.
The thing in [brackets] is the ratio of the total volume to the volume of the dissipation layer in the walls. It must be $\gg 1$ to give $Q \gg 1$.

Jackson also shows (8.99) that the frequency shift $\Delta\omega$ is always negative, and comparable in magnitude to the width:

$$\Delta\omega \approx \mathrm{Im}(\omega) \approx -\frac{\omega_0}{2Q}.$$

Related Topics (rest of Jackson chap. 8)

Schumann Resonances are resonances of the Earth-Ionosphere cavity. The Earth, at $r = R_E \sim 6400$ km, is one conducting "cap", and the ionosphere, at $r \approx R_E + 100$ km, is the other. The "side walls" do not exist physically but are replaced by the condition that all wave fields be periodic in longitude with period 2π.

Lightning creates EM noise over a wide range of frequencies. Waves near the Schumann resonance frequencies become temporarily trapped so the noise spectrum peaks near these frequencies (see Jackson Fig. 8.9).

The ionosphere is not a perfect conductor, and its skin depth is ~ its thickness, so the Q of this cavity is only mediocre (~ 10), but $\gtrsim 1$.

Optical Fibers are analogous to waveguides, except that the wave fields are confined, not by conducting walls, but by gradients of the index of refraction. The "core" of the fiber has larger n than the "walls" so wave fields are confined by total internal reflection.