PHYS 532 L20

Review of Waveguide theory (from last time):

Assume $\begin{Bmatrix} \underline{E} \\ \underline{B} \end{Bmatrix} \propto e^{i(kz-\omega t)}$ ($\hat{z}$ = axis of waveguide)

Then Ampere + Faraday $\Rightarrow$

$$\underline{E}_t = \frac{ik\nabla_t E_z - i\omega\,\hat{z}\times\nabla_t B_z}{\mu\epsilon\omega^2 - k^2} \qquad (8.26a)$$

$$\underline{B}_t = \frac{i\omega\mu\epsilon\,\hat{z}\times\nabla_t E_z + ik\nabla_t B_z}{\mu\epsilon\omega^2 - k^2} \qquad (8.26b)$$

And the wave equation becomes a 2-D eigenvalue equation

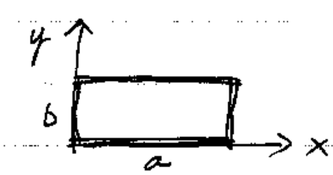
$$\left(\nabla_t^2 + \mu\epsilon\omega_\lambda^2\right)\begin{Bmatrix} E_{z0} \\ B_{z0} \end{Bmatrix} = 0 \qquad (8.34)$$

where $\omega_\lambda^2 = \omega^2 - \frac{k^2}{\mu\epsilon}$ are the eigenvalues determined by the boundary conditions

$$\begin{Bmatrix} E_{z0} \\ \frac{\partial B_{z0}}{\partial n} \end{Bmatrix} = 0 \text{ at surface of conductor,}$$

for $\begin{Bmatrix} TM \\ TE \end{Bmatrix}$ wave having $\begin{Bmatrix} B_{z0} \\ E_{z0} \end{Bmatrix} = 0$.

For a rectangular waveguide the eigenvalues are given by



$$\mu\epsilon\,\omega_\lambda^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \qquad (m, n = \text{integers})$$

For a TM mode, $B_{z0} \equiv 0$ and $E_{z0} = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$.

For a TE mode, $E_{z0} \equiv 0$ and $B_{z0} = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$.

For TM mode, the cut-off frequency (lowest frequency for a nontrivial eigenfunction) is given by $n = m = 1$, or $\mu\epsilon\,\omega_{11}^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$.

For TE mode, can have n or m (not both) $= 0$, so cut-off frequency is given by $n = 0$, $m = 1$ (assuming $a > b$), or $\mu\epsilon\,\omega_{1,0}^2 = \frac{\pi^2}{a^2}$.

It is generally true (not just for rectangular geometry) that the TE mode has a lower cut-off frequency than the TM mode.

<u>Example</u> (rectangular waveguide with $a = 2b$, $n = m = 1$)

<u>TM mode</u>: From (8.26),

$$\underline{E}_t = \frac{ik}{\mu\epsilon\omega_\lambda^2}\nabla_t E_{z0}$$

$$\underline{B}_t = \frac{\mu\epsilon\omega}{k}\,\hat{z}\times\underline{E}_t$$

$-E_{z0}(x,y)$ is a 2-D <u>potential</u> for $\underline{E}_t$, and $\underline{B}_t$ follows lines of constant potential.

$E_{z0} \propto \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b} \Rightarrow$

$$\underline{E}_t \propto \frac{\hat{x}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{\hat{y}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

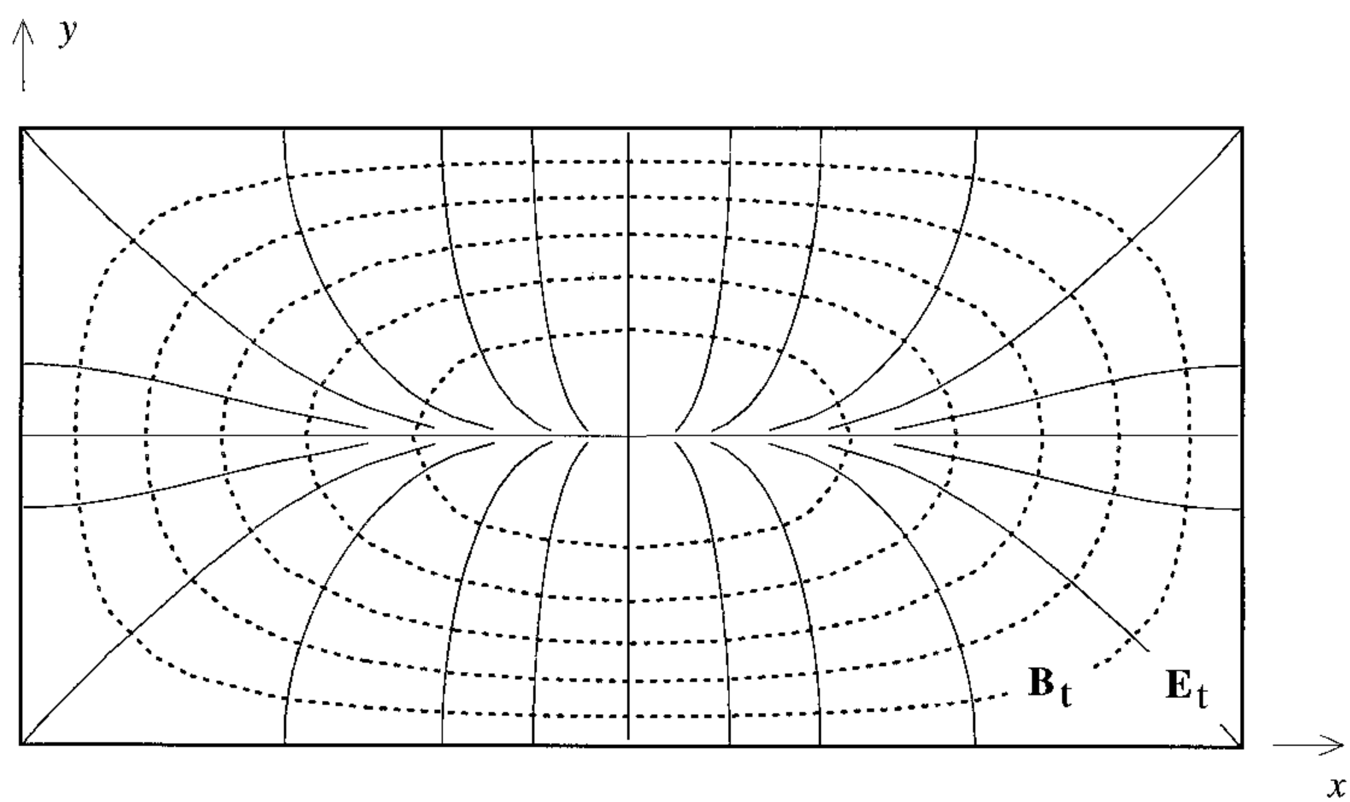
Lines of $\underline{E}_t$ given by $\frac{dy}{dx} = \frac{E_{ty}}{E_{tx}} = \frac{a\tan\frac{\pi x}{a}}{b\tan\frac{\pi y}{b}}$

or $\quad \tan\left(\frac{\pi y}{b}\right)\frac{dy}{dx} = \frac{a}{b}\tan\frac{\pi x}{a}$

which can be integrated to give

$$\log\left[\cos\left(\frac{\pi y}{b}\right)\right] = \frac{a^2}{b^2}\log\left[\cos\left(\frac{\pi x}{a}\right)\right] + \text{const.}$$

These lines of $\underline{E}_t$ are plotted as solid lines below. Dashed lines are lines of $\underline{B}_t$ = lines of const. E_{z0}.



TM (1,1) mode (a/b = 2)

<u>TE mode</u> : From (8.26), $\underline{B}_t = \frac{ik}{\mu\epsilon\omega_\lambda^2} \nabla_t B_{z0}$

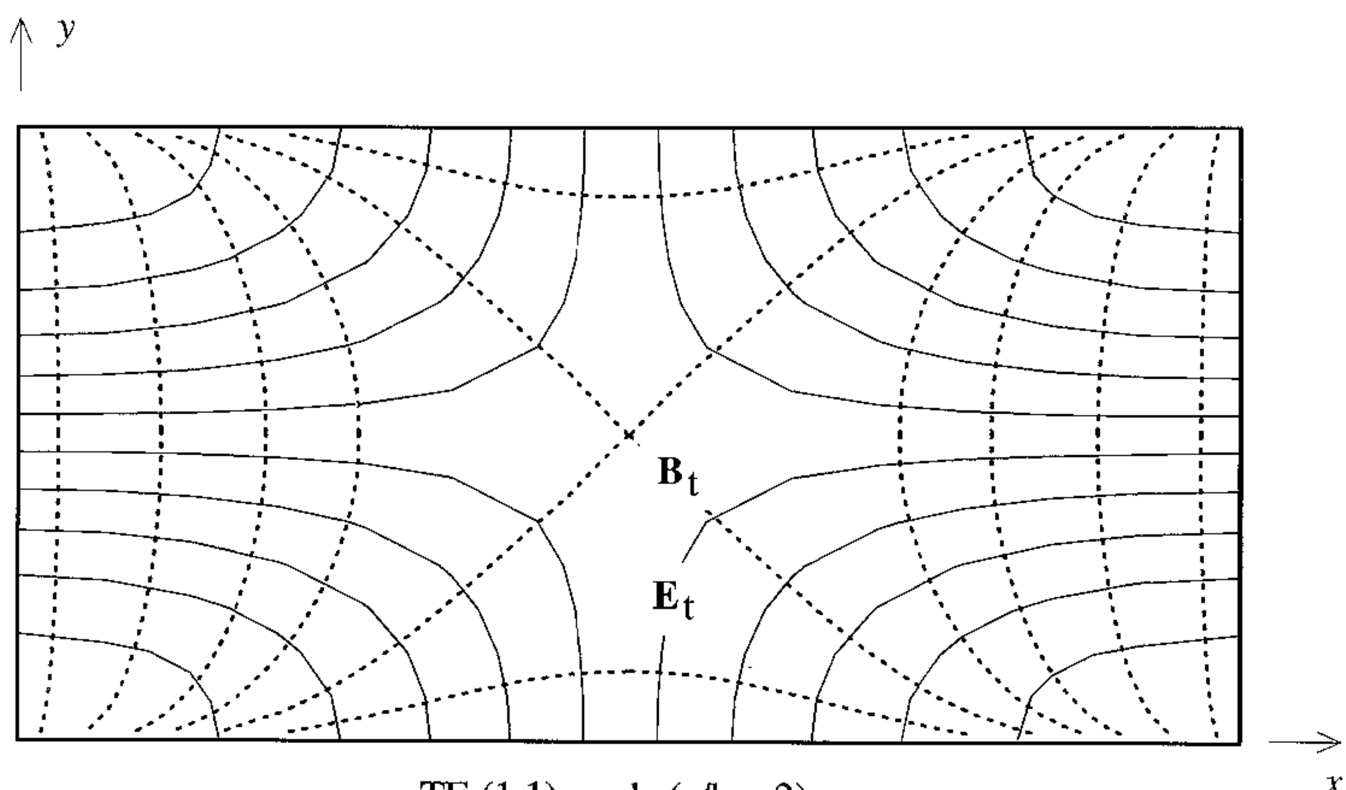
$$\underline{E}_t = \frac{-\omega}{k} \hat{z} \times \underline{B}_t$$

Now $B_{z0}(x,y)$ is a 2-D scalar potential for $\underline{B}_t$, and $\underline{E}_t$ follows lines of constant (magnetic) potential.

$$B_{z0} \propto \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \Rightarrow$$

$$\underline{B}_t \propto \frac{\hat{x}}{a} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{\hat{y}}{b} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

Lines of $\underline{B}_t$ plotted as dashed lines below.
Solid lines are lines of $\underline{E}_t$ = lines of constant B_{z0}.



TE (1,1) mode (a/b = 2)

8.5 Energy Flow and Attenuation in Waveguides

Power transmitted down the tube is $P = \int S_z \, da$ where $\underline{S}$ = Poynting vector, and the integral is over the cross-sectional area of tube.

$$S_z = \frac{1}{2} \mathrm{Re} \left(\underline{E}^* \times \underline{H} \right)_z = \frac{1}{2} \mathrm{Re} \left[\underline{E}_t^* \times \underline{H}_t \right] \cdot \hat{z}$$

For a TM mode, using (8.26) with $B_{z0} = 0$,

$$S_z = \frac{1}{2} \mathrm{Re} \left[\frac{-ik \nabla_t E_{z0}^*}{\mu \epsilon \omega_\lambda^2} \times \frac{i\omega\epsilon \hat{z} \times \nabla_t E_{z0}}{\mu\epsilon \omega_\lambda^2} \right] \cdot \hat{z}$$

$$= \frac{1}{2} \frac{\omega k}{\mu^2 \epsilon \omega_\lambda^4} \left| \nabla_t E_{z0} \right|^2 \qquad (\times 8.16)$$

For a TE mode, using (8.26) with $E_{z0} = 0$,

$$S_z = \frac{1}{2} \mathrm{Re} \left[\frac{i\omega \hat{z} \times \nabla_t B_{z0}}{\mu\epsilon\omega_\lambda^2} \times \frac{ik \nabla_t B_{z0}}{\mu^2 \epsilon \omega_\lambda^2} \right] \cdot \hat{z}$$

$$= \frac{1}{2} \frac{\omega k}{\mu^3 \epsilon^2 \omega_\lambda^4} \left| \nabla_t B_{z0} \right|^2 \qquad (\times 8.17)$$

In both cases, S_z is proportional to

$$\mathcal{I} \equiv \int \left| \nabla_t \psi \right|^2 da \quad \text{where}$$

$$\psi = \begin{Bmatrix} E_{z0} \\ B_{z0} \end{Bmatrix} \text{ for } \begin{Bmatrix} TM \\ TE \end{Bmatrix} \text{ mode.}$$

In general, $\mathcal{I} = \int \nabla_t \psi^* \cdot \nabla_t \psi \, da$

$$= \int \nabla_t \cdot (\psi^* \nabla_t \psi)\, da - \int \psi^* \nabla_t^2 \psi \, da$$

$$= \oint \hat{n} \cdot (\psi^* \nabla_t \psi)\, ds - \int \psi^* \nabla_t^2 \psi \, da \qquad (\text{x}8.18)$$

around circumference of waveguide, by Green's first identity (1.34).

For a TM mode, $\psi = E_{z0}$ vanishes on C.
For a TE mode, $\frac{\partial \psi}{\partial n} = \frac{\partial B_{z0}}{\partial n} = 0$ on C.

$\therefore$ First term in (x8.18) vanishes for both, and

$$\mathcal{I} = -\int \psi^* \nabla_t^2 \psi \, da$$

Wave equation (8.19) gives $\nabla_t^2 \psi = (k^2 - \mu\epsilon\omega^2)\psi = -\mu\epsilon\omega_\lambda^2 \psi$

so $\mathcal{I} = \mu\epsilon\omega_\lambda^2 \int \psi^* \psi \, da$

$$= \mu\epsilon\omega_\lambda^2 \int |\psi|^2 \, da$$

Using this in (8.16) and (8.17) gives

$$P = \frac{\omega k}{2\mu\omega_\lambda^2} \begin{cases} \int |E_{z0}^2| \, da & \text{TM mode} \\ \int \frac{1}{\mu\epsilon} |B_{z0}^2| \, da & \text{TE mode} \end{cases}$$

From the definition $\omega_\lambda^2 = \omega^2 - \frac{k^2}{\mu\epsilon}$ (p. 1),

$$k = \sqrt{\mu\epsilon}\,(\omega^2 - \omega_\lambda^2)^{1/2} = \sqrt{\mu\epsilon}\,\omega\left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}$$

and

$$P = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \begin{Bmatrix} \int \frac{\epsilon}{2}|E_{z0}|^2 da & TM \\ \int \frac{1}{2\mu}|B_{z0}|^2 da & TE \end{Bmatrix} \quad (8.51)$$

which is $\propto$ Energy density in $\begin{Bmatrix} E_{z0} \\ B_{z0} \end{Bmatrix}$ for $\begin{Bmatrix} TM \\ TE \end{Bmatrix}$ wave.

According to Jackson (8.52) it is straightforward to show that the energy density (per unit z) is

$$U = \left(\frac{\omega}{\omega_\lambda}\right)^2 \begin{Bmatrix} \int \frac{\epsilon}{2}|E_{z0}|^2 da & TM \\ \int \frac{1}{2\mu}|B_{z0}|^2 da & TE \end{Bmatrix}$$

so the ratio of energy flux to energy density is

$$\frac{P}{U} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = \frac{d\omega}{dk} = v_g \quad (8.53)$$

(v_g = group speed)

Note that $v_{ph} v_g = \frac{\omega}{k}\frac{d\omega}{dk} = \frac{1}{\mu\epsilon}$ (8.54)

The group speed is always $< \frac{1}{\sqrt{\mu\epsilon}}$ and $\to 0$ as $\omega \to \omega_\lambda$.

The power flow P is attenuated as a function of z by Ohmic dissipation in the walls of the waveguide, which is a constant rate of dissipation per unit length:

$$P(z) = P_0 \, e^{-2\beta_\lambda z} \qquad (8.56)$$

The attenuation length is $\frac{1}{2\beta_\lambda} \propto \sigma \delta(\omega_\lambda)$ (8.63)
where $\delta(\omega_\lambda)$ is the conductor skin depth at $\omega = \omega_\lambda$:

$$\delta(\omega_\lambda) = \sqrt{\frac{2}{\mu_c \omega_\lambda \sigma}} \qquad (8.8)$$

Thus attenuation length $\propto \sqrt{\sigma / \omega_\lambda}$

Efficiency $\Rightarrow$ large σ, small ω_λ.