PHYS 532 L2

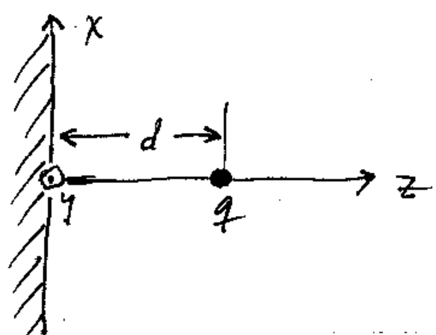
Jackson 2.1 - 2.5

Method of Images

Used to find the field of a point charge(s) in the presence of external boundary conditions. Choose the magnitude and position of the image charge such that its field, plus the field of the original charge, satisfies the boundary conditions.

Example 1 – Point charge and a grounded conducting plane.

"Grounded conducting plane" means it is held at a fixed potential – $\underline{E}$ is $\perp$ to it.



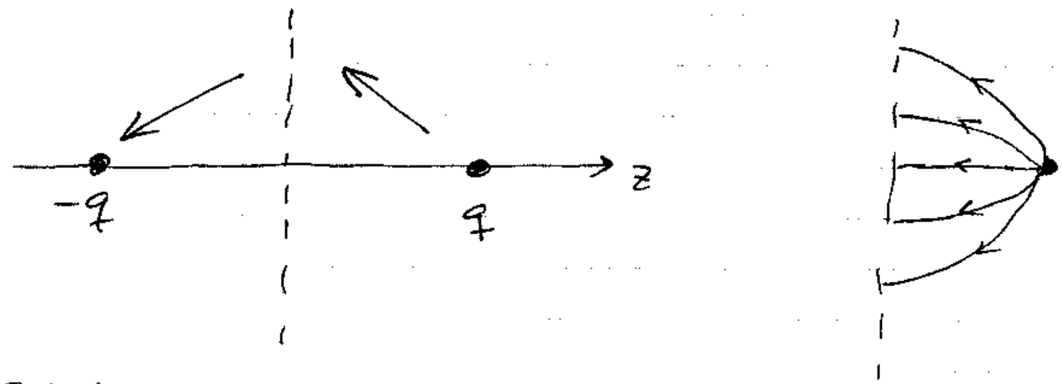
Find potential Φ for $z > 0$.

Potential eqn $\nabla^2 \Phi = \frac{-q}{\epsilon_0} \delta(x)\delta(y)\delta(z-d)$ (X2.1a)

for $z > 0$, with boundary conditions

$$\left.\begin{aligned} &\Phi = 0 \text{ at } z = 0 \\ &\Phi \to 0 \text{ as } \sqrt{x^2+y^2+z^2} \to \infty . \end{aligned}\right\} \quad (X2.1b)$$

We want the components E_x and E_y from the real and images charges to cancel, so we make the image charge equal in magnitude, opposite in sign, to the real charge, at a distance d on the other side of the plane:



Solution is

$$\underline{\Phi} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\underline{x} - d\hat{z}|} - \frac{1}{|\underline{x} + d\hat{z}|}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad (X2.2)$$

What is the charge density on the conducting plane?

$$\nabla \cdot \underline{E} = \frac{\partial E_z}{\partial z} = \rho/\epsilon_0 = \frac{\sigma(x,y)}{\epsilon_0}\,\delta(z)$$

$E = 0$ for $z < 0$. Integrate from $z = -\epsilon$ to ϵ:

$$E_z(x, y, \epsilon) = \frac{\sigma(x,y)}{\epsilon_0}$$

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$$\sigma(x,y) = -\lim_{\epsilon \to 0^+} \epsilon_0 \frac{\partial \Phi}{\partial z}(x,y,z)\Big|_{z=\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{2q}{4\pi} \frac{z-d}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}\Big|_{z=\epsilon}$$

$$= \frac{-2qd}{4\pi\,(x^2+y^2+d^2)^{3/2}} \qquad (X\,2.3)$$

Equipotentials :

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In[3]:= Phi[z_, y_] := 1 / Sqrt[y^2 + (z - 1) ^2] - 1 / Sqrt[y^2 + (z + 1) ^2]

In[7]:= ContourPlot[Phi[z, y], {z, 0, 4}, {y, -2, 2}, ContourShading → False, PlotPoints → 50]
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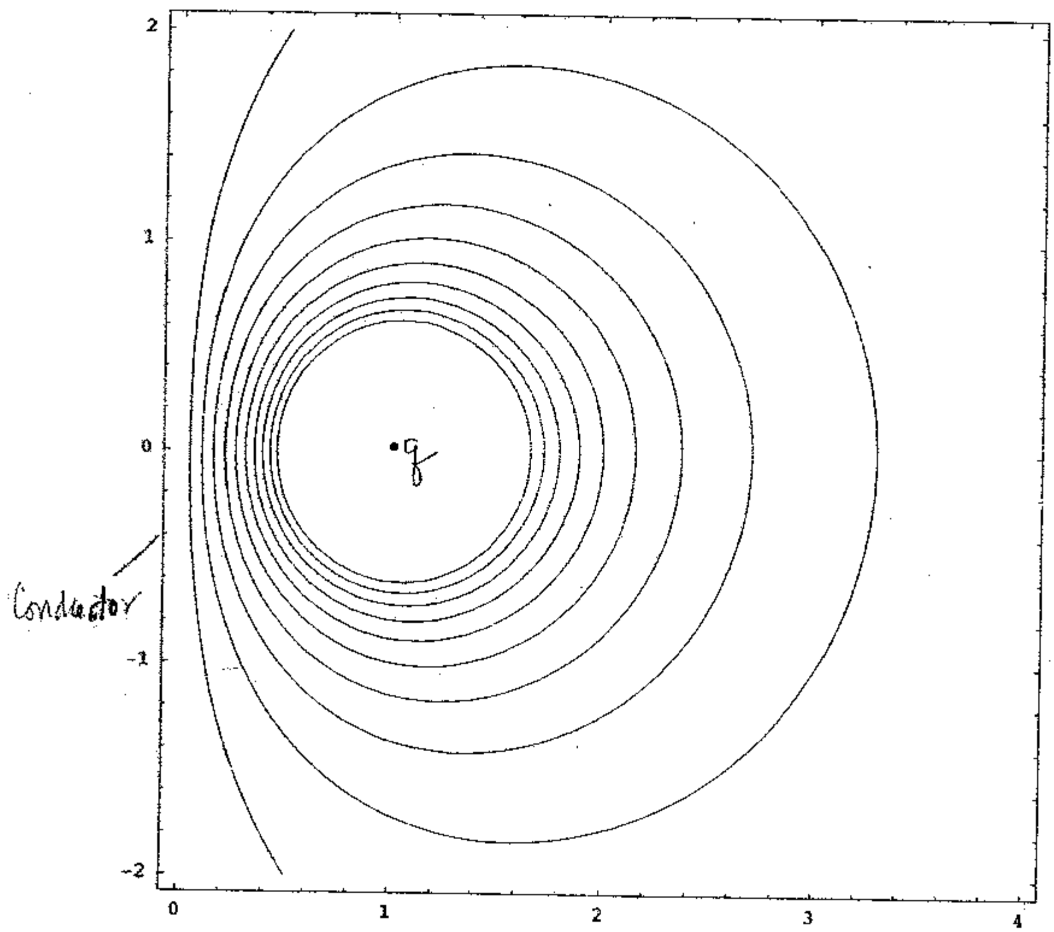


What is the force on the conducting wall?

Method 1: Using the surface-charge density calculated above:

$$F_z^{wall} = \int \sigma(x,y) E_z(x,y,\epsilon)\, dx\,dy = \int \frac{\sigma^2 dx\,dy}{\epsilon_0}$$

$$= \frac{1}{\epsilon_0} \int \left(\frac{2qd}{4\pi [x^2+y^2+d^2]^{3/2}} \right)^2 dx\,dy$$

$$= \frac{q^2 d^2}{4\pi^2 \epsilon_0} \underbrace{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(x^2+y^2+d^2)^3}}_{\underbrace{\int_0^\infty \frac{2\pi r\,dr}{(r^2+d^2)^3}}_{\pi \int_{d^2}^{\infty} \frac{du}{u^3} = \frac{\pi}{2d^4}}}$$

$(r = \sqrt{x^2+y^2})$

$(u = r^2+d^2)$

$$\boxed{F_z^{wall} = \frac{q^2}{8\pi \epsilon_0 d^2}}$$

Method 2: Calculate force on q from image method -

$$F_z^q = \frac{-q^2}{4\pi\epsilon_0 (2d)^2} = \frac{-q^2}{16\pi\epsilon_0 d^2}$$

Force on wall must be equal and opposite to the force on q:

$$F_z^{wall} = \frac{q^2}{16\pi \epsilon_0 d^2} \qquad (\times 2.4)$$

which is $\frac{1}{2}$ the force calculated by method 1.

Q: Which one is right?

A: Method 2 is right. The problem with method 1 is that E_z isn't constant through the charge layer.

$$F_z = \int \rho E_z \, dx\,dy\,dz = \int \epsilon_0 \frac{\partial E_z}{\partial z} E_z \, dx\,dy\,dz$$

$$= \frac{\epsilon_0}{2} \int \frac{\partial}{\partial z}(E_z^2) \, dx\,dy\,dz$$

$$= \frac{\epsilon_0}{2} \int \left[E_z(x,y,\epsilon)\right]^2 dx\,dy$$

$$= \frac{1}{2\epsilon_0} \int \sigma(x,y)^2 \, dx\,dy$$

$$\frac{\text{Force}}{\text{area}} \text{ on conductor } = \frac{\sigma^2}{2\epsilon_0} \quad \text{(general rule)}.$$

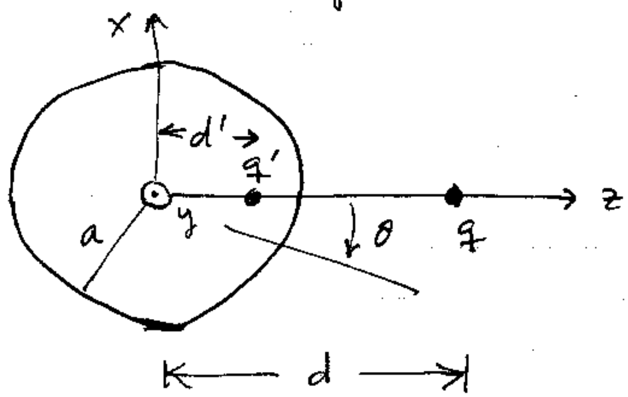
This is $\frac{1}{2}$ the earlier result from method 1, and = result from method 2.

The force is always attractive: $\frac{\text{Force}}{\text{area}} = \sigma \langle E_z \rangle$

where $\langle E_z \rangle$ = average E_z in charge layer.

What else can we do with the image charge method?

Example 2 : Point charge near a grounded conducting sphere.



Potential eqn $\nabla^2 \Phi = \frac{-q}{\epsilon_0} \delta(x)\delta(y)\delta(z-d)$

for $x^2+y^2+z^2 > a^2$

Boundary conditions $\Phi = \text{const}$ on $x^2+y^2+z^2 = a^2$

$\Phi \to 0$ for $x^2+y^2+z^2 \to \infty$

Assume there is an image charge solution with image charge q' at $z = d'$:

$$\underset{\text{outside}}{\Phi(x,y,z)} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-d')^2}}\right]$$

Use spherical coordinates

$z = r\cos\theta$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2-2dr\cos\theta+d^2}} + \frac{q'}{\sqrt{r^2-2d'r\cos\theta+d'^2}}\right]$$

Boundary Condition $\Phi(a,\theta,\phi)=0$ for all θ, ϕ

$$0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2-2ad\cos\theta+d^2}} + \frac{q'}{\sqrt{a^2-2ad'\cos\theta+d'^2}}\right]$$

Obviously q' has opposite sign to q.

For both terms to have same θ-dependence, we need

$$\frac{2ad}{a^2+d^2} = \frac{2ad'}{a^2+d'^2}$$

$$(a^2+d'^2)\,d = (a^2+d^2)\,d'$$

$$d\,d'^2 - (a^2+d^2)d' + a^2 d = 0$$

Solve the quadratic:

$$d' = \frac{a^2+d^2 \pm \sqrt{(a^2+d^2)^2-4a^2d^2}}{2d}$$

$$= \frac{a^2+d^2 \pm \sqrt{(d^2-a^2)^2}}{2d} = \left\{\begin{matrix} d \\ a^2/d \end{matrix}\right\}$$

The solution $d'=d$ is trivial; it gives $\Phi=0$ everywhere. The solution we want is

$$\boxed{d' = \frac{a^2}{d}} \qquad (2.4b)$$

$$\Phi(a,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2-2ad\cos\theta+d^2}} + \frac{q'}{\sqrt{a^2-2\frac{a^3}{d}\cos\theta+\frac{a^4}{d^4}}}\right]$$

$$\Phi(a,\theta,\varphi) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2-2ad\cos\theta+d^2}} + \frac{q'/q}{\frac{a}{d}\sqrt{d^2-2ad\cos\theta+a^2}}\right]$$

The two terms will cancel if $q' = -\frac{a}{d}q$ (2.4a)

The electric field is shown in Fig. 2.3 of Jackson.

Note: q' is the total physical charge induced on the grounded conducting sphere.

The same prescription works in reverse if we start with a point charge inside the sphere. In either case

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\underline{x}-d\hat{z}|} - \frac{a/d}{|\underline{x}-\frac{a^2}{d}\hat{z}|}\right] \quad (X2.8)$$

This also gives us the Green's function for the Laplace eqn. with conducting boundary conditions at $r=a$. Recall that $G(\underline{x},\underline{x}')$ is the potential at $\underline{x}$ due to a point charge $4\pi\epsilon_0$ at $\underline{x}'$. Thus, setting $q = 4\pi\epsilon_0$ and $\underline{x}' = d\hat{z}$ in (X2.8) gives

$$G_a(\underline{x},\underline{x}') = \frac{1}{|\underline{x}-\underline{x}'|} - \frac{a}{|\underline{x}'|\left|\underline{x}-\frac{a^2}{|\underline{x}'|^2}\underline{x}'\right|}$$

So if we have an arbitrary charge distribution $\rho(\underline{x}')$, with conducting b.c.'s at $r=a$, the potential is

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_{Vol} G_a(\underline{x},\underline{x}')\rho(\underline{x}')\,d^3x'$$