# Separation of Variables: Laplace Equation in Cylindrical Coordinates

- Laplace equation in cylindrical coordinates

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (3.71)$$

- Look for solution of the form

$$\Phi = R(\rho)\, Q(\phi)\, Z(z) \qquad (3.72)$$

- Equations for the three components:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad (3.73)$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \qquad (3.74)$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \qquad (3.75)$$

Bessel equation

- Solutions for $Z$ and $Q$ are simple:

$$Z(z) = e^{\pm kz}$$
$$Q(\phi) = e^{\pm i\nu\phi} \qquad (3.76)$$

- The solution to the radial equation (3.75) that is regular at the origin is

$$R = J_\nu(k\rho)$$

  - For $x \ll 1$, $J_\nu(x) \propto x^\nu$
  - For $x \gg 1$,

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

- There is a richer set of solutions that are irregular at the origin. A convenient choice is the Neumann function $N_n(k\rho)$:
  - For $x \ll 1$,

$$N_\nu(x) \propto x^{-\nu} \quad \text{for } \nu > 0$$

  - For $x \gg 1$,

$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

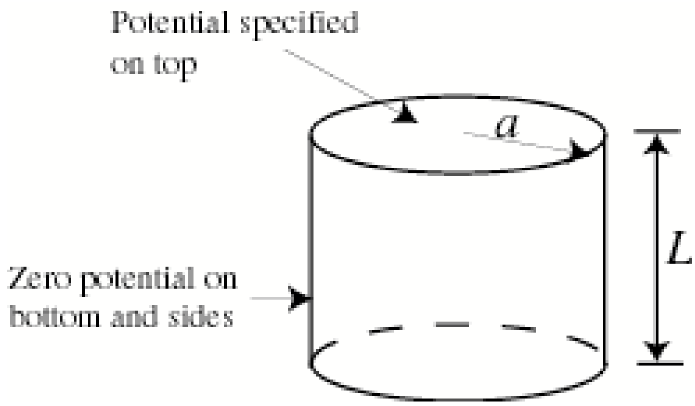
- Another convenient set of irregular solutions is the Hankel functions:

$$H_\nu^{(1,2)}(x) = J_\nu(x) \pm i\, N_\nu(x)$$

  - The two kinds of Hankel functions go like $e^{\pm ix} / |x|^{1/2}$ (Incoming and outgoing waves)

- If $k$ = imaginary, you get Bessel functions of imaginary arguments
  - Usually called $I_m$ and $K_m$
  - They are exponential at large distances instead of sinusoidal.

- Sample cylindrical boundary value problem:




$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z) (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

  - where

$$J_m(k_{mn}a) = 0$$

  - Solve for coefficients by enforcing $\Phi = V(\rho, \phi)$ on top.