Waveguides

Assume a hollow, cylindrical waveguide with perfectly conducting walls, cross-sectional shape arbitrary but independent of \( z \) (the axis of the cylinder):

\[ \downarrow \]

Assume \( E, B \propto e^{-i\omega t} \) inside, \( \rho = \varphi = 0 \) inside.

\[ \Rightarrow (\nabla^2 + \varepsilon \mu \omega^2) \left\{ \begin{array}{c} E \\ B \end{array} \right\} = 0 \quad \text{(as usual).} \quad (8.17) \]

Apply boundary conditions at the perfectly conducting walls: \( \mathbf{E} \times \mathbf{n} = 0 \), \( \mathbf{B} \cdot \mathbf{n} = 0 \) \( (8.1) \)

where \( \mathbf{n} \) is the outward normal unit vector.

If the cylinder has end caps, it's a resonant cavity (next time); otherwise, a waveguide. Because of the conducting cylinder, it's no longer necessary to have \( E_z = B_z = 0 \) (waves are not necessarily transverse).
Assume \( \left\{ \frac{E}{\beta} \right\} = \left\{ \frac{E}{\beta} (x, y) \right\} e^{-i k z - i \omega t} \) (\ref{8.18})

\[ \nabla_t^2 + (\epsilon \mu \omega^2 - k^2) \left\{ \frac{E}{\beta} \right\} = 0 \] (8.19)

where \( \nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \) is the transverse part of the gradient operator.

Separate \( E, \beta \) into parallel and transverse parts:

\[ E = E_z \hat{z} + E_t \quad \beta = B_z \hat{z} + B_t \]

Faraday:
\[ \nabla \times E = i \omega B \]

\[ \Rightarrow \hat{z} \cdot (\nabla_t \times E_t) = i \omega B_z \] (\ref{8.2})

\[ \nabla_t \times (E_z \hat{z}) + \nabla_t \times E_t = i \omega B_t \]

\[ \nabla_t E_z \times \hat{z} + i k \hat{z} \times E_t = i \omega B_t \]

\[ B_t = \frac{1}{i \omega} \nabla_t E_z \times \hat{z} - \frac{k}{i \omega} \hat{z} \times E_t \] (\ref{8.3})

Ampere
\[ \nabla \times B = -i \omega \mu_0 E \]

\[ \hat{z} \cdot \nabla_t \times B_t = -i \omega \mu_0 E_z \] (\ref{8.4})

\[ \nabla_t \times (B_z \hat{z}) + \nabla_t \times B_t = -i \omega \mu_0 E_t \]

\[ \nabla_t B_z \times \hat{z} + i k \hat{z} \times B_t = -i \omega \mu_0 E_t \]

crossing with \( \hat{z} \) from the left gives
\[ \nabla_t B_t = -i k B_t = -i \omega \mu e \hat{e} \times E_t \]

\[ B_t = \frac{\omega \mu e}{k} \hat{e} \times E_t - \frac{i}{k} \nabla_t B_t \quad (8.5) \]

Evaluating the right sides of (8.3) and (8.5),

\[ \frac{1}{i \omega} \nabla_t E_t \times \hat{e} + \frac{k}{\omega} \hat{e} \times E_t = \frac{\omega \mu e}{k} \hat{e} \times E_t - \frac{i}{k} \nabla_t B_t \]

Crossing with \( \hat{e} \) gives

\[ \frac{1}{i \omega} \nabla_t E_t - \frac{k}{\omega} E_t = -i \mu e \omega \hat{e} \times \nabla_t B_t \]

\[ E_t \left( -\frac{k}{\omega} + \frac{i \mu e \omega}{k} \right) = \frac{1}{i \omega} \nabla_t E_t - \frac{i}{k} \hat{e} \times \nabla_t B_t \]

\[ \times k \omega \rightarrow E_t \left( \mu e \omega^2 - k^2 \right) = i k \nabla_t E_t - i \omega \hat{e} \times \nabla_t B_t \]

\[ \Rightarrow E_t = \frac{i k \nabla_t E_t - i \omega \hat{e} \times \nabla_t B_t}{\mu e \omega^2 - k^2} \quad (8.26a) \]

Plugging this into (8.5) gives

\[ B_t = \frac{\omega \mu e}{k} \hat{e} \times \nabla_t E_t - \frac{i}{k} \hat{e} \times \left( \frac{2}{\mu e \omega^2 - k^2} \nabla_t B_t \right) \]

\[ = i \omega \mu e \hat{e} \times \nabla_t E_t + \frac{i \omega \mu e}{k} \nabla_t B_t - \frac{i}{k} \frac{(\mu e \omega^2 - k^2) \nabla_t B_t}{\mu e \omega^2 - k^2} \]

\[ B_t = \frac{i \omega \mu e \hat{e} \times \nabla_t E_t + i k \nabla_t B_t}{\mu e \omega^2 - k^2} \quad (8.26b) \]
Equations (8.26), though not very pretty, are rather remarkable. They say that if the parallel components \( E_z \) and \( B_z \) are given, the transverse components \( E_t \) and \( B_t \) can be derived from them. In particular, if \( E_z = B_z = 0 \), then \( E_t = B_t = 0 \), unless \( \mu \omega^2 = k^2 \).

This special case is the transverse electromagnetic (TEM) wave (later). First, let's look at two other special cases.

**Case I** \( E_z \neq 0, \quad B_z = 0 \)

Transverse Magnetic (TM) mode.

Set \( E_z = E_{z0}(x,y) e^{i(kz - \omega t)} \) \((x\,8.6)\)

with \( \frac{E_{z0}}{\text{(surface)}} = 0 \) \((x\,8.1)\)

From (8.26b), \( B_t = \frac{i\mu \omega E_z \times V_t E_{z0}}{\mu \omega^2 - k^2} \) \((x\,8.7)\)

\( E_{z0} = 0 \) at surface \( \Rightarrow V_t E_{z0} \parallel \hat{n} \), \( B_t \parallel \hat{n} = 0 \)

From (8.26a), \( E_t = \frac{i\omega E_{z0}}{\mu \omega^2 - k^2} \) \((x\,8.8)\)

\( E_t \parallel \hat{n} \) at surface.

Mode structure can be derived from (8.15) for a given geometry (eigenvalue problem).

\[ \nabla^2 E_{z0} = -\varepsilon \mu \omega^2 E_{z0} \] \((x\,8.9)\)

with \( \omega^2 = \frac{k^2}{\mu \varepsilon} + \omega_{\lambda}^2 \) \((x\,8.10)\)
Example: Rectangular waveguide

Try solution of the form

\[ E_z(x, y) = E_0 \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \]  \hspace{1cm} (8.11)

with \( m, n \) integers

Substitute into (8.19):

\[
-\frac{m^2 \pi^2}{a^2} - \frac{n^2 \pi^2}{b^2} - k^2 + \varepsilon \mu \omega^2 = 0
\]

\[
\omega^2 = \frac{1}{\varepsilon \mu} \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)
\]

Eigenvalues are \( \omega^2_{\lambda} = \frac{1}{\varepsilon \mu} \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \) \hspace{1cm} (8.12)

If either \( m \) or \( n \) is zero, \( E_z \equiv 0 \), so the lowest-frequency mode is

\[
\omega^2_{\lambda} \text{ (minimum)} = \frac{1}{\varepsilon \mu} \left( \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right) \hspace{1cm} (8.13)
\]

Case II. \( E_z = 0 \), \( B_z \neq 0 \) \hspace{1cm} (TM modes)

From (8.26a), \( E_t = \frac{-i \omega \hat{z} \times \nabla_t B_z}{\mu \varepsilon \omega^2 - k^2} \) \hspace{1cm} (8.14)

At surface, \( E_t \cdot \hat{n} = 0 \)

\[ \Rightarrow \hat{n} \times (\hat{z} \times \nabla_t B_z) = \hat{z} \left( \hat{n} \cdot \nabla_t B_z \right) = 0 \Rightarrow \frac{\partial B_z}{\partial n} = 0 \]
Example: Same rectangular waveguide as above:

Try  \( B_z = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{i(kz - \omega t)} \)

\[ \Rightarrow (\text{again}) \quad \omega^2 = \frac{k^2}{\mu \varepsilon} + \omega_0^2 \quad (x8.15a) \]

where  \( \omega_0^2 = \frac{1}{\mu \varepsilon} \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \quad (x8.15b) \)

But now we can have \( n \) or \( m = 0 \) (not both), so the cut-off frequency is given by

\( m = 1, \; n = 0 \) if \( a > b \) (and vice-versa).

Thus the lowest cut-off frequency for a given waveguide is the TE cut-off frequency.
(It also gives the most efficient energy transmission.)

Finally, the transverse electromagnetic (TEM) mode has  \( E_z = B_z = 0 \), which implies (8.26) that  \( k^2 = \mu \varepsilon \omega^2 \). This has a zero cut-off frequency (because  \( k \) is real for all  \( \omega \)), but it cannot propagate in a simply-connected cylindrical waveguide because (x8.1) and (x8.2) now give

\( \nabla \times E_t = 0 \Rightarrow E_t = -\nabla \Phi \quad \text{with} \quad \Phi = \text{constant on boundary} \Rightarrow \Phi = \text{constant inside} \).

It can, however, propagate in a multiply-connected waveguide, e.g., a two-wire transmission line \((0, 0)\) or a coaridal cable \(\Box\).