PHYS 532 Lecture 16

EM waves in a Conductor (continued) $(\omega \ll \omega_1)$

Last time we found that the effect of conductivity σ on a wave is included if we replace ϵ by $\epsilon + \frac{i\sigma}{\omega}$. If we try this in the basic dispersion relation $\frac{\omega^2}{k^2} = \frac{1}{\mu\epsilon}$, we get

$$\frac{\omega^2}{k^2} = \frac{1}{\mu(\epsilon + i\frac{\sigma}{\omega})} \qquad (\text{J } 7.66)$$

Clearly, both ω and k cannot be real. They could, of course, both be complex, but we will look at the two limiting cases (1) k real, ω complex, and (2) ω real, k complex.

(1) Assume k real, solve for $\omega(k)$
(Initial value problem)

$$\omega^2\left(\epsilon + i\frac{\sigma}{\omega}\right) = \frac{k^2}{\mu}$$

$$\omega^2 + \omega\frac{i\sigma}{\epsilon} - \frac{k^2}{\mu\epsilon} = 0$$

$$\omega = \frac{-i\sigma}{2\epsilon} \pm \sqrt{\frac{k^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2} \qquad (\text{J } 7.67)$$

The $\pm$ signs correspond to waves propagating in opposite directions.

Since $\mathrm{Im}(\omega) < 0$, and everything is $\propto e^{-i\omega t}$, both waves <u>decay</u> exponentially with time.

(A) In a <u>poor conductor</u>, with $\frac{\sigma}{2\epsilon} \ll \frac{k}{\sqrt{\mu\epsilon}}$,

$$\omega \approx \frac{-i\sigma}{2\epsilon} \pm \frac{k}{\sqrt{\mu\epsilon}}$$

The wave propagates at the usual phase speed $\left(\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}\right)$ but decays in time with the (long) time constant $\frac{2\epsilon}{\sigma} \gg 1/\omega$.

(B) In a <u>good conductor</u>, with $\frac{\sigma}{2\epsilon} \gg \frac{k}{\sqrt{\mu\epsilon}}$,

$$\omega = \frac{-i\sigma}{2\epsilon} \pm \frac{i\sigma}{2\epsilon}\sqrt{1 - \frac{4\epsilon k^2}{\mu\sigma^2}}$$

$$\approx \frac{-i\sigma}{2\epsilon} \pm \frac{i\sigma}{2\epsilon}\left(1 - \frac{2\epsilon k^2}{\mu\sigma^2}\right)$$

$$\approx \left.\begin{cases} -i\,\dfrac{k^2}{\mu\sigma} & \text{slow decay} \\ -i\,\dfrac{\sigma}{\epsilon} & \text{fast decay} \end{cases}\right\} \qquad (\times 7.68)$$

Let's look at the ratio E/B. Recall that $\nabla \times \underline{E} = \frac{-\partial B}{\partial t}$ so $|kE| = |\omega B|$ and

$$\left|\frac{E}{B}\right| = \left|\frac{\omega}{k}\right| = c \quad \text{in vacuum}$$

$$= \left\{\begin{matrix} \dfrac{k}{\mu\sigma} & \text{slow-decaying wave} \\ \dfrac{\sigma}{\epsilon k} & \text{fast-decaying wave} \end{matrix}\right\}$$

But $\quad \dfrac{\sigma}{2\epsilon} \gg \dfrac{k}{\sqrt{\mu\epsilon}} \quad \Rightarrow$

$$\frac{k}{\mu\sigma} \ll \frac{\sqrt{\mu\epsilon}\,\sigma}{2\mu\epsilon\sigma} = \frac{1}{2\sqrt{\mu\epsilon}} = \frac{c}{2} \quad \text{, and}$$

$$\frac{\sigma}{\epsilon k} \gg \frac{2}{\sqrt{\mu\epsilon}} = 2c \text{ ,}$$

so the fast-decaying wave has $\left|\frac{E}{B}\right| \gg c$ and the slow-decaying wave has $\left|\frac{E}{B}\right| \ll c$. After a while, $|E| \ll c|B|$ and the wave becomes a quasi-static magnetic field structure. In the limit $\sigma \to \infty$, the wave doesn't decay. <u>The magnetic field is frozen into a perfect conductor</u>.

For a good but not perfect conductor, the decay time is given by

$$\frac{1}{T} \sim \frac{k^2}{\mu\sigma} \sim \frac{1}{L^2\mu\sigma} \qquad (L \equiv 1/k = \text{length scale})$$

$$\frac{L^2}{T} = \frac{1}{\mu\sigma}$$ represents the diffusion rate of a magnetic-field structure.

Another way to see this:

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} = \sigma \underline{E} + \underbrace{\epsilon \frac{\partial \underline{E}}{\partial t}}_{\text{neglect}}$$

$$\nabla \times \nabla \times \underline{H} \simeq \sigma\, \nabla \times \underline{E}$$

$$\nabla(\nabla \cdot \underline{H}) - \nabla^2 \underline{H} = -\sigma \frac{\partial \underline{B}}{\partial t}$$

$$\nabla \cdot \underline{H} = 0 \;(\mu = \text{const}) \Rightarrow \frac{\partial \underline{B}}{\partial t} = \frac{1}{\mu\sigma} \nabla^2 \underline{B}$$

This is a <u>diffusion equation</u> for $\mathbf{B}$, with diffusion coefficient $D = \frac{1}{\mu\sigma}$ (Jackson § 5.18).

Note the amplitude of an EM wave always decays in a conductor, except
(a) $\sigma \to 0$ (non-decaying propagating wave),
or (b) $\sigma \to \infty$ (magnetic field frozen in).

(2) Now assume ω real, solve for k (complex):

$$k^2 = \omega^2 \mu \left(\epsilon + \frac{i\sigma}{\omega}\right) = \mu\epsilon\omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right)$$

This represents an EM wave, frequency ω, incident on a conductor. It decays as it propagates.

(A) <u>Poor conductor</u> ($\sigma/\epsilon\omega \ll 1$).

$$k \approx \pm\left(\sqrt{\mu\epsilon}\,\omega + \frac{i\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}\right)$$
$$= \pm\,(k_r + i k_i)$$

$\pm$ sign $\Rightarrow$ propagation in $\pm x$ direction.
Wave field $\propto \exp[ik_r x - k_i x - i\omega t]$.

Decay length $\frac{1}{k_i} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$ (x 7.69)

When we assumed k real for the poor conductor (p. 2), we found a decay time $1/\omega_i \sim 2\epsilon/\sigma$. Thus

$$\frac{\text{decay length}}{\text{decay time}} = \frac{\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}{\frac{2\epsilon}{\sigma}} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\approx \frac{\omega_r}{k_r} \approx \frac{d\omega_r}{dk_r} \qquad (\times 7.70)$$

= phase speed = group speed.
(No dispersion, just damping.)

(B) <u>Good conductor</u> ($\sigma/\epsilon\omega \gg 1$)

$$k^2 \approx i\mu\sigma\omega$$

$$k = \pm\sqrt{\mu\sigma\omega}\left(\frac{1+i}{\sqrt{2}}\right) = k_r + i k_i \qquad (5.164)$$

($\pm$ sign for propagation in $\pm x$ direction)

Note $k_i = k_r$ for this case — the wave amplitude declines by $e^{-2\pi} \approx 0.002$ in one wavelength.

The <u>skin depth</u> is $\delta = \frac{1}{k_i} = \sqrt{\frac{2}{\mu\sigma\omega}}$ (5.165)

= distance that a wave with frequency ω can penetrate a conductor of conductivity σ.

D. High-Frequency Limit, Plasma Frequency

Recall the general expression for $\epsilon(\omega)$

$$\epsilon = \epsilon_0 \left[1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \qquad (7.51)$$

The above discussion of conduction effects was based on the assumption $\omega << \omega_1$, the lowest resonance frequency of the material. Now let's make the opposite assumption $\omega >> \omega_j$ (all j) and also $\omega >> \gamma_j$. Then (7.51) gives

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2} \qquad (7.59)$$

where $$\omega_p^2 \equiv \frac{NZe^2}{\epsilon_0 m} \qquad (7.60)$$

NZ = electrons / unit volume.
ω_p = plasma frequency.

Combing (7.59) with the general dispersion relation (7.4) [or (x7.12) from L12] $k = \sqrt{\mu\epsilon}\,\omega$ gives the plasma dispersion relation

$$\omega^2 = \omega_p^2 + k^2 c^2 \qquad (7.61)$$

(assuming $\mu = \mu_0$).

In a dielectric, (7.61) applies only for $\omega^2 \gg \omega_p^2$. However, in a plasma, there are <u>only</u> free electrons (no bound electrons), and (7.61) applies also for $\omega < \omega_p$, where it implies k = imaginary (no propagation, only damping). (7.61) can be re-written

$$k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = i\,\frac{\omega}{c}\left(\frac{\omega_p^2}{\omega^2} - 1\right)^{1/2} \quad (\omega < \omega_p)$$

$$\approx i\,\frac{\omega_p}{c} \qquad (\omega \lesssim \omega_p)$$

The attenuation constant (absorption coefficient) is

$$\alpha \equiv 2\,\mathrm{Im}(k) = \frac{2\omega_p}{c}$$

$$\frac{c}{\omega_p} \equiv \underline{\text{electron skin depth}} = \delta_e$$

Near the ionospheric electron-density peak, $N \sim 10^6/\mathrm{cm}^3 = 10^{12}/\mathrm{m}^3 \Rightarrow \omega_p \sim 60$ MHz and $\delta_e \sim 5$ m $\ll$ scale height.
This is the basis of radio "sounding" of $N_e(z)$ as well as of the earth-ionosphere waveguide.

For metals, N (and hence ω_p) is much larger, and δ_e much smaller.