PHYS 532 lecture 15

Brewster's Angle

Last time we derived Fresnel formulas for $\underline{E} \perp$ plane of incidence (7.39) and for $\underline{E} \parallel$ plane of incidence (7.41). For $\parallel$ polarization, the reflected wave has

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - n\sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \qquad (7.41\,b)$$

There is an angle i_B for which $E_0'' = 0$, i.e., no reflected wave (with $\parallel$ polarization).

Setting the numerator of (7.41b) $= 0$,

$$\frac{\mu}{\mu'} n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$$

$$\left(\frac{\mu}{\mu'}\right)^2 n'^4 \left(1 - \sin^2 i_B\right) = n^2 n'^2 - n^4 \sin^2 i_B$$

$$\sin^2 i_B \left(n^4 - n'^4 \frac{\mu^2}{\mu'^2}\right) = n'^2 \left(n^2 - \frac{\mu^2}{\mu'^2} n'^2\right)$$

$$\sin^2 i_B = \frac{n'^2 \left(n^2 - \frac{\mu^2}{\mu'^2} n'^2\right)}{n^4 - n'^4 \frac{\mu^2}{\mu'^2}}$$

In the usual situation, where $\mu \approx \mu' \approx \mu_0$,

$$\sin^2 i_B \approx \frac{n'^2 (n^2 - n'^2)}{n^4 - n'^4} = \frac{n'^2}{n^2 + n'^2} < 1$$

$\therefore$ i_B real and $\boxed{\tan i_B = \frac{n'}{n}}$ (7.43)

So, at $i = i_B$, the $\parallel$-polarized wave is 100% transmitted (no reflection), whereas the $\perp$-polarized wave is still partially reflected.

This is one way to produce polarized light — reflect it from a dielectric at Brewster's angle.

Moving on to Jackson § 7.5 —

Frequency Dispersion Characteristics

A. Simple model for $\epsilon(\omega)$

Assume the electrons in a medium are held in place by "springs" (harmonic-oscillator potentials with natural frequency ω_0) and a little bit of friction:

$$m\ddot{\underline{x}} = -e\underline{E} - m\omega_0^2 \underline{x} - m\underbrace{\gamma}_{\text{friction coefficient}}\dot{\underline{x}}$$

(The magnetic force is neglected.)

Assume $\underline{x} \propto e^{-i\omega t}$

$$\Rightarrow \quad -m\omega^2 \underline{x} = -e\underline{E} - m\omega_0^2 \underline{x} + mi\omega\gamma \underline{x}$$

$$\underline{x}\left[m(\omega_0^2-\omega^2) - im\gamma\omega\right] = -e\underline{E}$$

$$\underline{x} = \frac{-e}{m(\omega_0^2-\omega^2-i\gamma\omega)}\underline{E} .$$

Each electron produces a dipole moment $\underline{p} = -e\underline{x}$ so the polarization vector is

$$\underline{P} = \frac{e^2 \underline{E}}{m} N \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

where N = number density of molecules and f_j = number of electrons / molecule having natural frequency ω_j (oscillator strength).

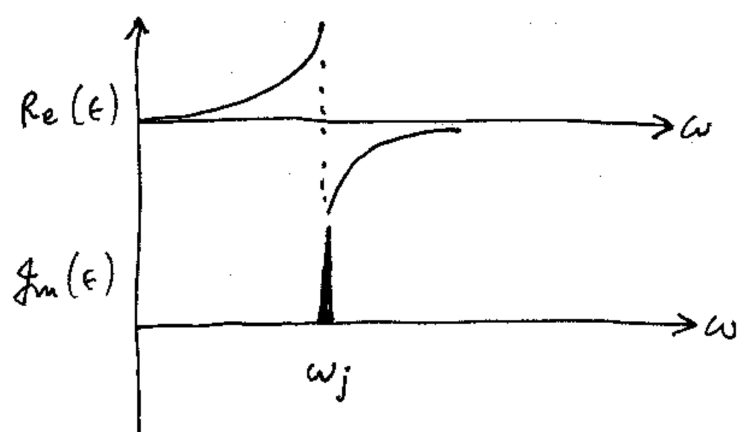
Using the definitions $\underline{P} = \chi \epsilon_0 \underline{E}$ (4.36) and $\epsilon = \epsilon_0 (1+\chi)$ (4.38), this gives

$$\epsilon = \epsilon_0 \left[1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right] \quad (7.51)$$

This actually works for real atoms, given proper quantum-mechanical definitions of f_j, γ_j, ω_j.

B. Anomalous Dispersion & Resonant Absorption

For $\gamma_j = 0$, the resonances ($\omega \to \omega_j$) in (7.51) would look like



For $|\gamma_j| \ll \omega_j$ (but not zero), the peaks are large but not infinite (like Jackson Fig. 7.8).

$$\frac{d\,Re(\epsilon)}{d\omega} > 0 \Rightarrow \text{"\underline{normal dispersion}"}$$

$$\frac{d\,Re(\epsilon)}{d\omega} < 0 \Rightarrow \text{"\underline{anomalous dispersion}"}$$

The cross-over, where $Im(\epsilon)$ is large, is the region of "<u>resonant absorption</u>".

Assume the wave-vector magnitude k is complex:

$$k = \beta + i\frac{\alpha}{2} \tag{7.53}$$

where α = "attenuation constant" or "absorption coefficient".

$$\rightarrow \quad k^2 = \beta^2 - \frac{\alpha^2}{4} + i\alpha\beta$$

Comparing this with the dispersion relation

$$k^2 = \mu_0 \epsilon \omega^2 \qquad (\mu = \mu_0) \tag{7.4}$$

$$\left.\begin{aligned} &\text{gives} \quad \beta^2 - \frac{\alpha^2}{4} = \mu_0\omega^2\,Re(\epsilon) = \frac{\omega^2}{c^2}Re\left(\frac{\epsilon}{\epsilon_0}\right) \\ &\text{and} \quad \alpha\beta = \mu_0\omega^2\,Im(\epsilon) = \frac{\omega^2}{c^2}Im\left(\frac{\epsilon}{\epsilon_0}\right) \end{aligned}\right\} \tag{7.54}$$

If we know $\epsilon(\omega)$ (a property of the medium), we can get the <u>dispersion relation</u> $\omega = \omega(k)$.

C. Low-frequency Behavior, Electrical Conductivity

A conductor has $\omega_0 \approx 0$ for its lowest natural frequency. ("free" electrons) Leaving out the ω_j terms for $j > 0$, we can write (7.51) as

$$\epsilon \approx \epsilon_0 \left[1 - \frac{Ne^2}{\epsilon_0 m} \frac{f_0}{\omega(\omega + i\gamma_0)} \right] \qquad (\omega \ll \omega_1)$$

$$= \epsilon_0 \left[1 + \frac{i}{\omega} \frac{Ne^2 f_0}{\epsilon_0 m(\gamma_0 - i\omega)} \right] \qquad (X7.61)$$

We can obtain an alternate form of $\epsilon(\omega)$ for a conductor by combining <u>Ohm's law</u>

$$\underline{J} = \sigma \underline{E} \qquad (X7.62)$$

with Ampere's law

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} = \underline{J} + \epsilon \frac{\partial \underline{E}}{\partial t} \qquad (X7.63)$$

If $\underline{E} \propto e^{-i\omega t}$, (X7.63) becomes

$$\nabla \times \underline{H} = -i\omega\left(\epsilon + \frac{i\sigma}{\omega}\right) \underline{E} \qquad (X7.64)$$

instead of the $\underline{J} = 0$ form $\nabla \times \underline{H} = -i\omega\epsilon \underline{E}$.

$\therefore$ The effect of conductivity can be included if we simply replace ϵ by $\epsilon + \frac{i\sigma}{\omega}$.

Comparing (X7.64) with (X7.61), we see that the two approaches are consistent, <u>provided</u>

$$\sigma(\omega) = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \qquad (7.58)$$

This is, in fact, the standard (classical) model for conductivity, with $f_0 N$ representing the free-electron number density, and γ_0 the electron momentum-exchange collision frequency.

The steady-state $(\omega = 0)$ conductivity is just

$$\sigma(\omega = 0) = \frac{f_0 N e^2}{m \gamma_0} \qquad (\times 7.65)$$

Note that $\mathrm{Re}\left[\sigma(\omega)\right] = \frac{f_0 N e^2}{m} \frac{\gamma_0}{\gamma_0^2 + \omega^2}$

$$\rightarrow 0 \text{ for } \omega \gg \gamma_0$$

and $\mathrm{Im}\left[\sigma(\omega)\right] = \frac{f_0 N e^2}{m} \frac{\omega}{\gamma_0^2 + \omega^2}$

$$\rightarrow 0 \text{ for } \omega \ll \gamma_0$$