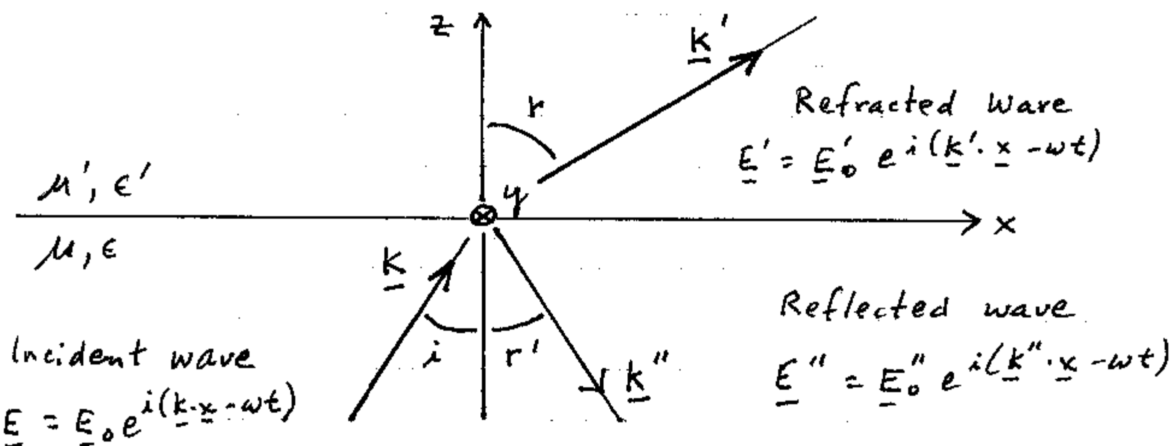
Reflection & Refraction

Consider a plane interface between two media



Choose axes such that $k_y = 0$.
(xz plane is the "plane of incidence" defined by the wave vector $\underline{k}$ and the surface normal $\hat{z}$.)

To keep all 3 waves in phase on the surface,

$$k_x = k_x' = k_x'' \quad \text{and} \quad k_y' = k_y'' = 0$$

$$k''^2 = k^2 \quad \text{and} \quad k_x'' = k_x \Rightarrow k_z'' = -k_z$$

$$\Rightarrow \boxed{r' = i} \qquad (\times 7.28)$$

(Angle of reflection = angle of incidence.)

$$k'^2 = \mu'\epsilon'\omega^2 = k_x'^2 + k_z'^2 = \mu\epsilon\omega^2\sin^2 i + \mu'\epsilon'\omega^2\cos^2 r \qquad (\times 7.29)$$

Defining $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$, $n' = \sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}$, (x7.30)

(x7.29) becomes $n'^2 = n^2\sin^2 i + n'^2\cos^2 r$

$$\Rightarrow \boxed{n'\sin r = n\sin i} \quad \text{(Snell's Law)} \qquad (7.36)$$

To find what fraction of the wave is transmitted and what fraction reflected, apply boundary conditions at surface:

$$\Delta E_{\text{tangential}} = 0 \Rightarrow \begin{aligned} E_{oy} + E''_{oy} &= E'_{oy} \quad &(x7.31)\\ E_{ox} + E''_{ox} &= E'_{ox} \quad &(x7.32)\end{aligned}$$

$$\Delta H_{\text{tangential}} = 0 \Rightarrow \begin{aligned} H_{oy} + H''_{oy} &= H'_{oy} \quad &(x7.33)\\ H_{ox} + H''_{ox} &= H'_{ox} \quad &(x7.34)\end{aligned}$$

and Faraday's Law $\Rightarrow \underline{B} = \underline{k}\times\underline{E}/\omega$ (x7.35)

Case 1. $\underline{E} \parallel \hat{y}$ (polarization $\perp$ plane of incidence)

From (x7.35), $B_x = \frac{-k_z E_y}{\omega} \Rightarrow H_x = \frac{-k_z E_y}{\mu\omega}$ (x7.36)

similarly $H''_x = \frac{-k''_z E''_y}{\mu\omega}$

But $k''_z = -k_z \Rightarrow H''_x = \frac{k_z E''_y}{\mu\epsilon}$ (x7.37)

Similarly $H'_x = \frac{-k'_z E'_y}{\mu'\omega}$ (x7.38)

Substituting (x7.36) - (x7.38) in (x7.34) $\rightarrow$

$$\frac{-k_z E_{oy}}{\mu\omega} + \frac{k_z E''_{oy}}{\mu\omega} = -\frac{k'_z E'_{oy}}{\mu'\omega} \qquad (x7.39)$$

Multiplying (x7.39) by $\omega\mu\mu'$ and re-arranging gives

$$\mu k_z' E_{0y}' + \mu' k_z E_{0y}'' = \mu' k_z E_{0y} \qquad (x7.40)$$

Rewriting (x7.31) gives

$$E_{0y}' - E_{0y}'' = E_{0y} \qquad (x7.41)$$

Solving (x7.40) and (x7.41) gives

$$\frac{E_{0y}'}{E_{0y}} = \frac{\begin{vmatrix} \mu' k_z & \mu' k_z \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} \mu k_z' & \mu' k_z \\ 1 & -1 \end{vmatrix}} = \frac{2\mu' k_z}{\mu k_z' + \mu' k_z} \qquad (x7.42)$$

$$\frac{E_{0y}''}{E_{0y}} = \frac{-\begin{vmatrix} \mu k_z' & \mu' k_z \\ 1 & 1 \end{vmatrix}}{\mu k_z' + \mu' k_z} = \frac{\mu' k_z - \mu k_z'}{\mu k_z' + \mu' k_z} \qquad (x7.43)$$

But (x7.29) gives $\dfrac{k_z'}{k_z} = \sqrt{\dfrac{k'^2 - k_x^2}{k^2 - k_x^2}}$

$$= \sqrt{\frac{\frac{n'^2\omega^2}{c^2} - \frac{n^2\omega^2}{c^2}\sin^2 i}{\frac{n^2\omega^2}{c^2} - \frac{n^2\omega^2}{c^2}\sin^2 i}}$$

$$= \sqrt{\frac{n'^2}{n^2} - \sin^2 i} \Big/ \cos i \qquad (x7.44)$$

Substituting (x7.44) in (x7.42) and (x7.43) gives ...

$$\left[\begin{aligned} \frac{E'_{0y}}{E_{0y}} &= \frac{2\mu'}{\frac{\mu}{\cos i}\sqrt{\frac{n'^2}{n^2}-\sin^2 i}+\mu'} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}} && (7.39a) \\ \frac{E''_{0y}}{E_0} &= \frac{\mu' - \frac{\mu}{\cos i}\sqrt{\frac{n'^2}{n^2}-\sin^2 i}}{\frac{\mu}{\cos i}\sqrt{\frac{n'^2}{n^2}-\sin^2 i}+\mu'} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}} && (7.39b) \end{aligned}\right.$$

Called the Fresnel formulas for $\underline{E}_0 \perp$ plane of incidence.

Are the Fresnel formulas consistent with conservation of energy? i.e., is $S_z + S_z'' = S_z'$?

$$\langle S_z \rangle = \frac{1}{2}\,\mathrm{Re}\left[\hat{z}\cdot(\underline{E}\times\underline{H}^*)\right] \qquad (7.47)$$

$$\left.\begin{aligned} &(\mathrm{X}7.36) \rightarrow \langle S_z \rangle = \frac{1}{2}\mathrm{Re}\left[-E_{0y}H_{0x}^*\right] = \frac{|E_{0y}|^2 k_z}{2\mu\omega} \\ &(\mathrm{X}7.38) \rightarrow \langle S_z' \rangle = \frac{1}{2}\mathrm{Re}\left[-E'_{0y}H'^*_{0x}\right] = \frac{|E'_{0y}|^2\,\mathrm{Re}(k'_z)}{2\mu'\omega} \\ &(\mathrm{X}7.37) \rightarrow \langle S_z'' \rangle = \frac{1}{2}\mathrm{Re}\left[-E''_{0y}H''^*_{0x}\right] = \frac{-|E''_{0y}|^2 k_z}{2\mu\omega} \end{aligned}\right\} (\mathrm{X}7.45)$$

$$\langle S_z + S_z'' - S_z' \rangle = \frac{|E_{0y}|^2 k_z}{2\mu\omega}\left[1 - \left|\frac{E''_{0y}}{E_{0y}}\right|^2 - \frac{\mu k'_z}{\mu' k_z}\left|\frac{E'_{0y}}{E_{0y}}\right|^2\right] \quad (\mathrm{X}7.46)$$

$$[\;] = 1 - \left|\frac{n\cos i - \frac{\mu}{\mu'}\sqrt{\;}}{n\cos i + \frac{\mu}{\mu'}\sqrt{\;}}\right|^2 - \frac{\mu\,\mathrm{Re}\sqrt{\;}}{\mu' n\cos i}\left|\frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{\;}}\right|^2$$

$$= \frac{\left|n\cos i + \frac{\mu}{\mu'}\sqrt{\;}\right|^2 - \left|n\cos i - \frac{\mu}{\mu'}\sqrt{\;}\right|^2 - \frac{\mu}{\mu'}4n\cos i\,\mathrm{Re}\sqrt{\;}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$= 0.$$

Poynting flux in = Poynting flux out.
Energy is conserved.

The fraction of the energy flux that is transmitted is

$$T = \frac{S_z'}{S_z} = \frac{4\mu n \cos i \, \mathrm{Re}\sqrt{n'^2 - n^2\sin^2 i}}{\mu'\left(n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}\right)^2}$$

$\equiv$ <u>transmission coefficient</u> (X7.47)

and the fraction reflected is

$$R = \frac{-S_z''}{S_z} = \left| \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \right|^2$$

$\equiv$ <u>reflection coefficient</u> (X7.48)

Energy conservation $\Rightarrow T + R = 1$ (X7.49)

What if $n'^2 < n^2 \sin^2 i$?
Then the square root is imaginary, $R = 1$ and $T = 0$.
In this case, Snell's law ($n^2\sin^2 i = n'^2 \sin^2 r$) can't be satisfied for any real angle r.
There is no refracted wave.

This is called <u>total internal reflection</u>.

The wave is trapped in the region of larger n.

From (X7.44), $\frac{k_z'}{k_z} = \frac{\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i}$ = Imaginary.

Imaginary k_z' $\Rightarrow$ refracted wave is

<u>evanescent</u> (decays exponentially with depth).

The reflected wave has all the energy, but it does suffer a phase shift according to (7.39b).

Case 2. $\underline{E} \perp \hat{y}$ (polarization $\parallel$ plane of incidence)

Now we use the other pair of boundary conditions:

$$E_{0x} + E_{0x}'' = E_{0x}' \qquad (x7.32)$$

$$H_{0y} + H_{0y}'' = H_{0y}' \qquad (x7.33)$$

and the same propagation condition

$$\underline{B} = \underline{k} \times \underline{E} / \omega \qquad (x7.35)$$

$$\Rightarrow \underline{H} = \underline{k} \times \underline{E} / \mu\omega \qquad (x7.50)$$

Applying (x7.50) to each wave gives

$$H_{0y} = \frac{k_z E_{0x} - k_x E_{0z}}{\mu\omega} = \frac{-k E_0}{\mu\omega} \qquad (x7.51)$$

$$H_{0y}'' = -\frac{k E_0''}{\mu\omega} \qquad (x7.52)$$

$$H_{0y}' = -\frac{k' E_0'}{\mu'\omega} \qquad (x7.53)$$

Then (x7.33) gives

$$\frac{-k E_0}{\mu\omega} - \frac{k E_0''}{\mu\omega} = -\frac{k' E_0'}{\mu'\omega} \qquad (x7.54)$$

(x7.32) $\Rightarrow$ $$-E_0\cos i + E_0''\cos i = -E_0'\cos r \qquad (x7.55)$$

(x7.54) and (x7.55) can be rewritten

$$\frac{\mu k'}{\mu' k} E_0' - E_0'' = E_0 \qquad (x7.56)$$

$$\cos r\, E_0' + \cos i\, E_0'' = \cos i\, E_0 \qquad (x7.57)$$

Solving gives

$$\frac{E_0'}{E_0} = \frac{\begin{vmatrix} 1 & -1 \\ \cos i & \cos i \end{vmatrix}}{\begin{vmatrix} \frac{\mu k'}{\mu' k} & -1 \\ \cos r & \cos i \end{vmatrix}} = \frac{2\cos i}{\frac{\mu k'}{\mu' k}\cos i + \cos r} \qquad (x7.58)$$

From Snell's law ($n'\sin r = n\sin i$),

$$\cos r = \sqrt{1 - \frac{n^2}{n'^2}\sin^2 i} \qquad (x7.59)$$

$$\frac{k'}{k} = \frac{n'}{n} \qquad (x7.60)$$

$\rightarrow$

$$\left.\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \\ \text{Similarly} \quad \frac{E_0''}{E_0} &= \frac{\frac{\mu}{\mu'} n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'} n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \end{aligned}\right\} \quad (7.41)$$

the <u>Fresnel formulas</u> for polarization in the plane of incidence.