Polarization of a Plane Wave

$$\underline{E}(\underline{x},t) = \underline{E}_0\, e^{i(\underline{k}\cdot\underline{x}-\omega t)} \qquad (\times 7.22)$$

$$\underline{E}_0 = E_1 \hat{\underline{\epsilon}}_1 + E_2 \hat{\underline{\epsilon}}_2$$

Where $\hat{\underline{\epsilon}}_1$ and $\hat{\underline{\epsilon}}_2$ are unit vectors, orthogonal to each other and to $\underline{k}$: $\hat{\underline{\epsilon}}_1 \times \hat{\underline{\epsilon}}_2 = \hat{\underline{k}}$.

let $E_1 = a_1 e^{i\delta_1}$, $E_2 = a_2 e^{i\delta_2}$ $\qquad (\times 7.23)$

where $a_{1,2}$ and $\delta_{1,2}$ are real constants, $a_{1,2} > 0$.

The nature of the wave depends on the relative phase $\delta_2 - \delta_1$.

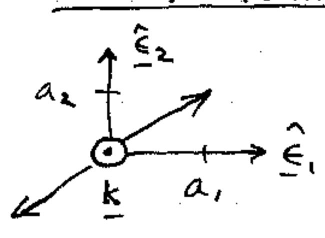
Case 1. $\delta_2 = \delta_1 + n\pi$ (n = integer)

$$\underline{E} = \left[a_1 \hat{\underline{\epsilon}}_1 + a_2 (-1)^n \hat{\underline{\epsilon}}_2\right] \exp\left[i(\underline{k}\cdot\underline{x} - \omega t + \delta_1)\right]$$

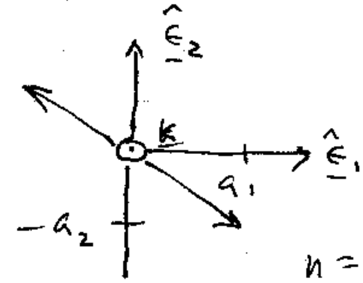
$$\underline{E}_p(\underline{x}=0,t) = \mathrm{Re}\left[\underline{E}(\underline{x}=0,t)\right]$$

$$= \left[a_1 \hat{e}_1 + a_2 (-1)^n \hat{\underline{\epsilon}}_2\right] \cos(\omega t - \delta_1)$$

$\Rightarrow$ Linear Polarization :



n = even



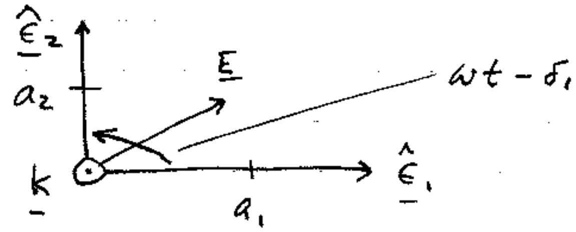
n = odd

These are the same (directions of $\hat{\underline{\epsilon}}_{1,2}$ arbitrary).

<u>Case 2</u>. $\delta_2 - \delta_1 = \pi/2$

$$\underline{E} = \left[a_1 \hat{\underline{\epsilon}}_1 + i a_2 \hat{\underline{\epsilon}}_2\right] \exp\left[i(\underline{k}\cdot\underline{x} - \omega t + \delta_1\right]$$

$$\underline{E}_p(\underline{x}=0,t) = a_1 \hat{\underline{\epsilon}}_1 \cos(\omega t - \delta_1) + a_2 \hat{\underline{\epsilon}}_2 \sin(\omega t - \delta_1)$$



$\underline{E}$ rotates counterclockwise, as viewed by an observer looking into the wave.

If $a_1 = a_2$, $\underline{E}$ traces out a circle, and the counterclockwise rotation is called <u>left circular polarization</u>.

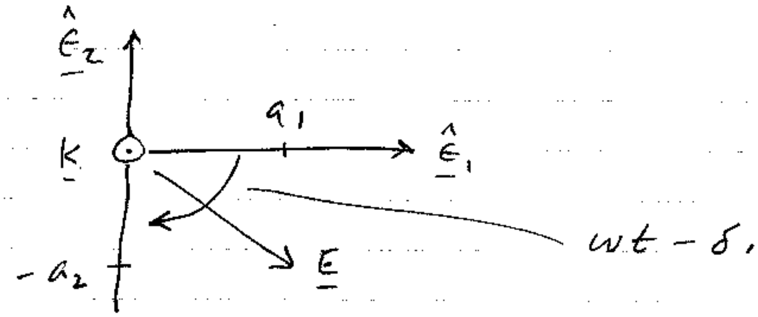
If $a_1 \neq a_2$, $\underline{E}$ traces out an ellipse; this is called <u>left elliptical polarization</u>.

This is also called <u>positive helicity</u>. ($\underline{\omega}$ of the $\underline{E}$ vector is $\parallel$ $\underline{k}$.)

<u>Case 3</u>. $\delta_2 - \delta_1 = -\pi/2$

$$\underline{E} = \left[a_1 \hat{\underline{\epsilon}}_1 - i a_2 \hat{\underline{\epsilon}}_2\right] \exp\left[i(\underline{k}\cdot\underline{x} - \omega t + \delta_1)\right]$$

$$\underline{E}_p(\underline{x}=0,t) = a_1 \hat{\underline{\epsilon}}_1 \cos(\omega t - \delta_1) - a_2 \hat{\underline{\epsilon}}_2 \sin(\omega t - \delta_1)$$



This is called <u>right elliptical polarization</u> or <u>negative helicity</u>.

For the general case (arbitrary $\delta_2 - \delta_1$), $\underline{E}$ traces out a tilted ellipse.
The ratio a_2/a_1 determines the height-to-width ratio of the ellipse.
The relative phase $\delta_2 - \delta_1$ determines its ellipticity, from a straight line for $\delta_2 - \delta_1 = n\pi$, to a ratio of minor/major axis $= \min\left(\frac{a_2}{a_1}, \frac{a_1}{a_2}\right)$ for $\delta_2 - \delta_1 = \pm\pi/2$.

$\therefore$ The amplitude and polarization of a plane EM wave at frequency ω and direction $\hat{\underline{k}}$ are characterized by 3 numbers, $a_1, a_2, \delta_2 - \delta_1$.

(The phase δ_1 is not measurable in classical optics.)

Another basic property is the <u>degree of polarization</u>.

A <u>monochromatic</u> <u>plane wave</u> is always 100% polarized (linear or elliptical).

However, there is <u>no such thing</u> as a monochromatic plane wave in nature.

There is always a finite length to any wave train, and hence a mixture of different frequency components that may be close to, but not equal to, ω. (More on that later.)

A finite wave train can be, at most, partially polarized. The percent of polarization gives an indication of how much variation there is of a_2/a_1 and $\delta_2 - \delta_1$ among the different spectral components.

In classical optics, the amplitude and polarization of a non-monochromatic wave is characterized by four <u>Stokes parameters</u>:

$$\left.\begin{aligned} s_0 &= \langle a_1^2 \rangle + \langle a_2^2 \rangle \\ s_1 &= \langle a_1^2 \rangle - \langle a_2^2 \rangle \\ s_2 &= \langle 2 a_1 a_2 \cos(\delta_2 - \delta_1) \rangle \\ s_3 &= \langle 2 a_1 a_2 \sin(\delta_2 - \delta_1) \rangle \end{aligned}\right\} \quad (7.27)$$

Note that these are not all independent because they depend only on 3 parameters $(a_1, a_2, \delta_2 - \delta_1)$.

$$s_1^2 + s_2^2 + s_3^2 = \langle a_1^2 \rangle^2 - 2\langle a_1^2 \rangle \langle a_2^2 \rangle + \langle a_2^2 \rangle^2$$
$$+ 4\left[\langle a_1 a_2 \cos(\delta_2 - \delta_1) \rangle^2 + \langle a_1 a_2 \sin(\delta_2 - \delta_1) \rangle^2\right]$$

If a_2/a_1 and $\delta_2 - \delta_1$ are constant, the waves are completely polarized and

$$s_1^2 + s_2^2 + s_3^2 = s_0^2 \qquad (X7.24)$$

For completely unpolarized light,

$$\langle a_1 a_2 \cos(\delta_2 - \delta_1) \rangle = \langle a_1 a_2 \sin(\delta_2 - \delta_1) \rangle = 0$$

because the phases of the different components are random, and

$$\langle a_1^2 \rangle = \langle a_2^2 \rangle$$

because there is no preferred direction, so

$$s_1^2 + s_2^2 + s_3^2 = 0 \quad \text{(unpolarized)}.$$

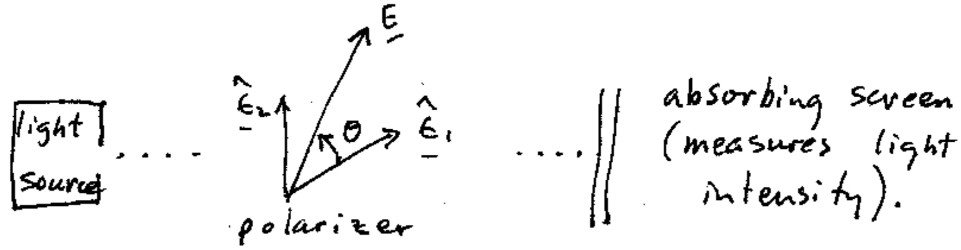
It is conventional to define the

$$\underline{\text{degree of polarization}} = \frac{s_1^2 + s_2^2 + s_3^2}{s_0^2} \qquad (X7.25)$$

which ranges from 0 (unpolarized) to 1 (completely polarized).

All four Stokes parameters can be measured using (1) a detector (measures energy flux),
(2) a polarizer (transmits only waves polarized in one direction), and
(3) a "quarter-wave plate" (which phase lags one polarization by $\pi/2$ relative to the other).

The first 3 can be measured using only a polarizer and a detector.



$$\langle |\underline{E}|^2 \rangle_{screen} = \left\langle \left| (E_1 \hat{\underline{\epsilon}}_1 + E_2 \hat{\underline{\epsilon}}_2) \cdot (\cos\theta\, \hat{\underline{\epsilon}}_1 + \sin\theta\, \hat{\underline{\epsilon}}_2) \right|^2 \right\rangle$$

$$= \langle | E_1 \cos\theta + E_2 \sin\theta |^2 \rangle$$

$$= |E_1|^2 \cos^2\theta + |E_2|^2 \sin^2\theta + 2 \mathrm{Re}\,(E_1^* E_2) \cos\theta \sin\theta$$

Using (X7.23) this becomes

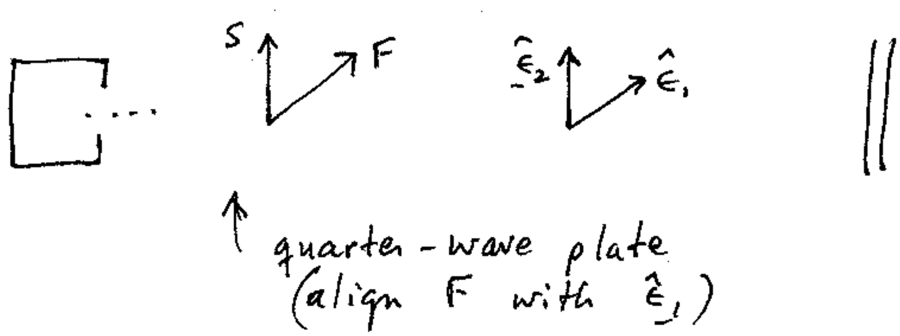
$$\langle |\underline{E}|^2 \rangle_{screen} = \langle a_1^2 \rangle \cos^2\theta + \langle a_2^2 \rangle \sin^2\theta$$

$$+ 2 \langle a_1 a_2 \cos(\delta_2 - \delta_1) \rangle \sin\theta \cos\theta$$

$$= \frac{1}{2} \left(s_0 + s_1 \cos 2\theta + s_2 \sin 2\theta \right) \qquad (X7.26)$$

By measuring $\langle |\underline{E}|^2 \rangle_{screen}$ at 3 angles, one can measure s_0, s_1, and s_2.

To measure s_3, add a quarter-wave plate, with thickness h, between the source and the polarizer:



The wave polarized $\parallel F$ propagates faster, with wave number k_F.

The wave polarized $\parallel S$ propagates slower, with wave number k_S.

Such that $(k_S - k_F)h = \pi/2$.

The wave entering the quarter-wave plate has

$$\underline{E} = \left(a_1 e^{i\delta_1} \hat{\underline{\epsilon}}_1 + a_2 e^{i\delta_2} \hat{\underline{\epsilon}}_2\right) e^{i\sim}$$

The wave leaving the quarter-wave plate has

$$\underline{E} = \left(a_1 e^{i\delta_1} \hat{\underline{\epsilon}}_1 + i a_2 e^{i\delta_2} \hat{\underline{\epsilon}}_2\right) e^{i\sim}$$

so $\langle |\underline{E}|^2 \rangle_{screen} = \Big\langle \Big| \left(a_1 e^{i\delta_1} \hat{\underline{\epsilon}}_1 + i a_2 e^{i\delta_2} \hat{\underline{\epsilon}}_2\right) \cdot \left(\cos\theta\, \hat{\underline{\epsilon}}_1 + \sin\theta\, \hat{\underline{\epsilon}}_2\right) \Big|^2 \Big\rangle$

$$= \langle |a_1 e^{i\delta_1} \cos\theta + i a_2 e^{i\delta_2} \sin\theta|^2 \rangle$$

$$= \langle |a_1^2| \rangle \cos^2\theta + \langle |a_2^2| \rangle \sin^2\theta + 2\cos\theta\sin\theta \underbrace{\mathrm{Re}\langle i a_1 a_2 e^{i(\delta_2 - \delta_1)} \rangle}_{-\langle a_1 a_2 \sin(\delta_2 - \delta_1) \rangle}$$

$$= \frac{1}{2}\left(s_0 + s_1 \cos 2\theta - s_3 \sin 2\theta\right) \qquad (\times 7.27)$$

which provides a meaure of s_3 (given s_0 and s_1).