Chapter 1 Summary

- Definition of Electric Field
  - To find the electric field at **x**, place an infinitely small test charge at **x** and measure the force on it:

$$\mathbf{F} = q\mathbf{E}(\mathbf{x}) \tag{1.1}$$

- Coulomb's Law
  - Field due to a collection of point charges:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_o} \sum_{n=1}^{\infty} q_i \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3} \tag{1.4}$$

    - **E** is radially out from a positive point charge, radially in toward a negative one.
    - Proportional to the -2 power of the distance from the point charge
  - Field due to a continuous charge distribution:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_o} \int \rho(\mathbf{x}') \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \tag{1.5}$$

- SI Units
  - $\varepsilon_o = 8.854 \times 10^{-12}$ Farad/m
  - **E** is in V/m, charge is in Coulombs (C)
  - $e = 1.6 \times 10^{-19}$ C = charge of proton
  - Jackson uses SI units for the first 10 chapters, Gaussian cgs units for chapters 11-16.

- Gauss's Law

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \frac{1}{\varepsilon_o} \sum_{i \text{ inside S}} q_i \tag{1.10}$$

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \frac{1}{\varepsilon_o} \int_V \rho(\mathbf{x})\, d^3x \tag{1.11}$$

  - The total flux of electric field emanating from a charge q is q/ε_o.
- Differential equations governing electrostatic fields:
  - Applying (1.11) to an arbitrarily small volume and using the divergence theorem it is easy to show that

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_o \tag{1.13}$$

  - (1.5) implies that

$$\nabla \times \mathbf{E} = 0 \tag{1.14}$$

  - Applying Stokes' theorem leads to

$$\oint \mathbf{E} \cdot \mathbf{dl} = 0 \tag{1.21}$$

- Electrostatic Potential
  - Any curl-free field can be represented as the gradient of a scalar:

$$\mathbf{E} = -\nabla\Phi \qquad (1.16)$$

  - One potential that is consistent with (1.5) and (1.16) is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{|\mathbf{x}-\mathbf{x}_i|} \text{ or } \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \qquad (1.17)$$

  - Nonuniqueness:
    - (1.17) is not the only potential that is consistent with (1.5) and (1.16). Any constant could be added to (1.17) and it would be equally consistent.
  - Differential equation for the potential
    - Combining (1.13) and (1.16) gives

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_o} \qquad \text{(Poisson's eqn.)} \qquad (1.28)$$

    - For the special case of a charge-free region (e.g., vacuum), Poisson's equation reduces to

$$\nabla^2\Phi = 0 \qquad \text{(Laplace's eqn.)} \qquad (1.29)$$

- Delta Functions
  - The potential at **x** due to a point charge q at $\mathbf{x}_1$:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_o} \, \frac{1}{\left|\mathbf{x} - \mathbf{x}_1\right|} \qquad \text{(X1.1)}$$

  - If the charge distribution is continuous,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad (1.17)$$

  - The usual way to include (X1.1) in (1.17) is by setting the charge density due to a point charge q to

$$\rho(\mathbf{x}') = q \;\; \delta^{(3)}(\mathbf{x}' - \mathbf{x}_1) \qquad \text{(X1.2)}$$

    where $\delta^{(3)}(\mathbf{x}'-\mathbf{x}_1) = 0$ if $\mathbf{x}' \neq \mathbf{x}_1$ and its volume integral over all space is 1.

$$\delta(x-a) = 0 \text{ for } x \neq a \tag{X1.3}$$

$$\int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)\, dx = 1 \tag{X1.4}$$

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta(x-a)\, dx = f(a) \tag{X1.5}$$

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta'(x-a)\, dx = -f'(a) \tag{X1.6}$$

$$\delta\left[f(x)\right] = \sum_{n=1}^{\infty} \frac{\delta(x-x_\mathrm{i})}{\left|df(x_\mathrm{i})/dx\right|} \tag{X1.7}$$

Note: Units of $\delta(g)$ = units of $1/g$

3D Delta functions:

Cartesian coordinates:

$$\delta^{(3)}(\mathbf{x}-\mathbf{X}) = \delta(x-X)\, \delta(y-Y)\, \delta(z-Z) \tag{X1.8}$$

$$\iiint \delta^{(3)}(\mathbf{x}-\mathbf{X})\, dx\, dy\, dz = 1 \tag{X1.9}$$

Cylindrical coordinates:

$$\delta^{(3)}(\mathbf{x}-\mathbf{X}) = \frac{\delta(r-R)\, \delta(\phi-\Phi)\, \delta(z-Z)}{r} \tag{X1.10}$$

$$\iiint \delta^{(3)}(\mathbf{x}-\mathbf{X})\, r\, dr\, d\phi\, dz = 1 \tag{X1.11}$$

Spherical coordinates:

$$\delta^{(3)}(\mathbf{x}-\mathbf{X}) = \frac{\delta(r-R)\, \delta(\theta-\Theta)\, \delta(\phi-\Phi)}{r^2 \sin\theta} \tag{X1.12}$$

$$\iiint \delta^{(3)}(\mathbf{x}-\mathbf{X})\, r^2\, dr\, \sin\theta\, d\theta\, d\phi = 1 \tag{X1.13}$$

- Green's Functions in Electrostatics
  - Write (1.28) for the case of a point charge at $\mathbf{x}_1$:

$$\nabla^2 \Phi = -4\pi\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_1) \qquad \text{(X1.14)}$$

  - The solution is given by (X1.1).
- The potential due to a point charge $4\pi\varepsilon_o$ is called the "Green's function." It satisfies

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}') \qquad (1.39)$$

And apparently a Green's function for Poisson's equation is

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \qquad \text{(X1.15)}$$

or

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\,\delta^{(3)}(\mathbf{x}-\mathbf{x}') \qquad (1.31)$$

- Q: Is (X1.15) the only solution to (1.39)?
  - A: No, you could add to (X1.15) any solution of Laplace's equation, and it would still satisfy (1.39), i.e.,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x}) \qquad (1.40)$$

where

$$\nabla^2 F = 0 \qquad (1.41)$$

  - Laplace's equation has many Green's functions. We normally choose a specific one by specifying boundary conditions.
    - The simple form (X1.15) is the one for which G goes to zero for large $|\mathbf{x}|$.