Chapter 9 Integration on $\mathbb{R}^n$

This chapter represents quite a shift in our thinking. It will appear for a while that we have completely strayed from calculus. In fact, we will not be able to "calculate" anything at all until we get the fundamental theorem of calculus in Section E and the Fubini theorem in Section F. At that time we'll have at our disposal the tremendous tools of differentiation and integration and we shall be able to perform prodigious feats.

In fact, the rest of the book has this interplay of "differential calculus" and "integral calculus" as the underlying theme. The climax will come in Chapter 12 when it all comes together with our tremendous knowledge of manifolds. Exciting things are ahead, and it won't take long!

In the preceding chapter we have gained some familiarity with the concept of n-dimensional volume. All our examples, however, were restricted to the simple shapes of n-dimensional parallelograms. We did not discuss volume in anything like a systematic way; instead we chose the elementary "definition" of the volume of a parallelogram as base times altitude. What we need to do now is provide a systematic way to think about volume, so that our "definition" of Chapter 8 actually becomes a theorem.

We shall in fact accomplish much more than a good discussion of volume. We shall define the concept of integration on $\mathbb{R}^n$. Our goal is to analyze a function

$$D \xrightarrow{f} \mathbb{R},$$

where D is a bounded subset of $\mathbb{R}^n$ and f is bounded. We shall try to define the *integral of f over D*, a *number* which we shall denote

$$\int_D f.$$

In the language of one-variable calculus, we are going to be defining a so-called *definite* integral. We shall never even think about extending the idea of *indefinite* integral to $\mathbb{R}^n$.

We pause to explain our choice of notation. Eventually we shall perform lots of calculations of integrals, and we shall then use all sorts of notations, such as

$$\int_D f(x)dx,$$

$$\int_D f(x_1, \ldots, x_n)dx_1 \ \ldots \ dx_n,$$

$$\int \cdots \int_D f(x_1, \ldots, x_n)dx_1 \ \ldots \ dx_n,$$

and more. But during the time we are developing the theory of integration, it would be inconvenient to use an abundance of notation. Therefore we have chosen the briefest notation that includes the function f and the set D and an integration symbol,

$$\int\limits_D f.$$

Analogy: in a beginning study of Latin the verb amare (to love) is used as a model for learning the endings of that declension, rather than some long word such as postulare. And in Greek a standard model verb is $\lambda\upsilon\omega$ (to loose).

In the special case $f = 1$, we obtain the expression for n-dimensional volume,

$$\text{vol}(D) = \text{vol}_n(D) = \int\limits_D 1.$$

A. **The idea of Riemann sums**

Bernhard Riemann was not the first to define the concept of a definite integral. However, he was the first to apply a definition of integration to any function, without first specifying what properties the function has. He then singled out those functions to which the integration process assigned a well defined number. These functions are called "integrable." He then derived a necessary and sufficient condition for a function to be integrable. This is contained in his *Habilitationsschrift* at Göttingen, 1854, entitled "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe."

We shall at first be working with *rectangular* parallelograms in $\mathbb{R}^n$, those parallelograms whose edges are mutually orthogonal. Actually, we shall be even more restrictive at first, and consider only those whose edges are in the directions of the coordinate axes. We call them *special rectangles*. Each of these may be expressed as a Cartesian product of intervals in $\mathbb{R}$:

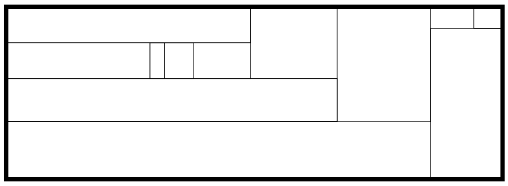
$$\begin{aligned} I &= [a_1, b_1] \times \cdots \times [a_n, b_n] \\ &= \{x \mid x \in \mathbb{R}^n,\ a_i \le x_i \le b_i \quad \text{for} \quad 1 \le i \le n\}. \end{aligned}$$

We then define the volume of I by analogy with our experience for $n = 1, 2, 3$:

$$\text{vol}(I) = (b_1 - a_1) \cdots (b_n - a_n).$$

This number was denoted $\text{vol}_n(I)$ in Chapter 8. We dispense with the subscript, as the dimension of all our rectangles in the current discussion is always that of the ambient space $\mathbb{R}^n$. Of course in dimension 1 the number $\text{vol}(I)$ is the length of I and in dimension 2 it is the area of I. But it is best to have a single term, "volume," to handle all dimensions simultaneously.

We next define a *partition* of I to be a collection of non-overlapping special rectangles I_1, $I_2, \dots, I_N$ whose union is I. "Non-overlapping" requires that the *interiors* of these rectangles are mutually disjoint. Notice that we certainly cannot require that the I_j's themselves be disjoint.



If f is a real-valued function defined on I, we then say that a *Riemann sum* for f is any expression of the form

$$\sum_{j=1}^{N} f(p_j)\text{vol}(I_j),$$

where each p_j is any point in I_j.

The brilliant idea of Riemann is to declare that f is *integrable* if these sums have a limiting value as the lengths of the edges of the rectangles in a partition tend to zero.

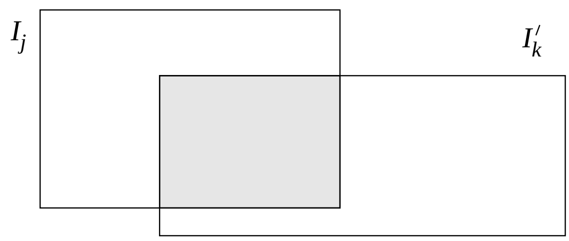
All the above is the precise analogy of the case $n = 1$, which is thoroughly discussed in one-variable calculus courses.

B. **Step functions**

Rather than follow the procedure of analyzing Riemann sums, we shall outline an equivalent development first published by Gaston Darboux in 1875. A very convenient notation for the sums involved relies on the elementary concept of the integral of a step function.

DEFINITION. Let I be a special rectangle, and let $I_1, I_2, \dots, I_N$ constitute a partition of I. A function $I \xrightarrow{\varphi} \mathbb{R}$ is called a *step* function if φ is constant on the interior of each I_j.

Notice that the collection of step functions on I enjoys many elementary algebraic properties. One of the most interesting is that the sum and product of two step functions is again a step function. Here are the details. Suppose φ corresponds to the partition $I_1, I_2, \dots, I_N$ of I, and another step function φ' corresponds to another partition $I'_1, I'_2, \dots, I'_{N'}$. Then a new partition can be formed by utilizing all the intersections (some of which may be empty) $I_j \cap I'_k$.



Clearly, $\varphi + \varphi'$ and $\varphi\varphi'$ are both constant on the interior of each $I_j \cap I'_k$, and thus they are step functions.

Notice that we say nothing about the behavior of a step function at points of the boundary of I_j. The reason for this apparent oversight is that the boundary consists of $(n-1)$-dimensional rectangles, which have zero volume (in $\mathbb{R}^n$) and thus will contribute nothing to our integral.

Now we simply write down a sum that corresponds to the Riemann sums we spoke of in Section A:

DEFINITION. Let $I \xrightarrow{\varphi} \mathbb{R}$ be a step function. Suppose that a corresponding partition of I consists of $I_1, I_2, \dots, I_N$, and that φ takes the value c_j on the interior of I_j. Then the *integral* of φ is the number

$$\int_I \varphi = \sum_{j=1}^{N} c_j \operatorname{vol}(I_j).$$

PROBLEM 9–1. There is a fine point in the above definition. It is that a given step function φ can be presented in terms of many choices of partitions of I. Thus the definition of the integral of φ might perhaps depend on the choice of partition. The essential fact that eliminates this difficulty is that for any partition of I,

$$\operatorname{vol}(I) = \sum_{j=1}^{N} \operatorname{vol}(I_j).$$

Prove this fact.

REMARK. You may think the above problem is silly, requiring a proof of an obvious statement. I would tend to agree, except that we should at least worry a little about our intuition for volumes in $\mathbb{R}^n$ for $n \geq 4$. Having said that, I am not interested in seeing a detailed proof of the problem, unless you can give a really elegant one.

Now we can rather easily prove that the integral of a step function is indeed well defined. Suppose that we have two partitions

$$I = \bigcup_{j=1}^{N} I_j = \bigcup_{k=1}^{N'} I'_k,$$

and that

$$\begin{aligned} \varphi &= c_j \quad \text{on the interior of } I_j, \\ \varphi &= c'_k \quad \text{on the interior of } I'_k. \end{aligned}$$

Then Problem 9–1 implies

$$\begin{aligned} \sum_j c_j \operatorname{vol}(I_j) &= \sum_j c_j \sum_k \operatorname{vol}(I_j \cap I'_k) \\ &= \sum_{j,k} c_j \operatorname{vol}(I_j \cap I'_k). \end{aligned}$$

In this double sum we need be concerned only with terms for which $\text{vol}(I_j \cap I'_k) \neq 0$. For such terms the rectangles I_j and I'_k have interiors which meet at some point p; thus $\varphi(p) = c_j$ and $\varphi(p) = c'_k$. Thus $c_j = c'_k$ for all terms that matter. Thus we obtain

$$\begin{aligned} \sum_{j,k} c_j \text{vol}(I_j \cap I'_k) &= \sum_{j,k} c'_k \text{vol}(I_j \cap I'_k) \\ &= \sum_k c'_k \sum_j \text{vol}(I_j \cap I'_k) \\ &= \sum_k c'_k \text{vol}(I'_k), \end{aligned}$$

the latter equality being another application of Problem 9–1. We conclude that

$$\sum_j c_j \text{vol}(I_j) = \sum_k c'_k \text{vol}(I'_k),$$

and thus $\int_I \varphi$ is well-defined.

There are just a few properties of these integrals that we need, and they are now easily proved.

THEOREM. *For step functions on* I *the following properties are valid:*

$$\begin{aligned} \int_I (\varphi_1 + \varphi_2) &= \int_I \varphi_1 + \int_I \varphi_2, \\ \int_I c\varphi &= c \int_I \varphi \quad \textit{if } c \textit{ is a constant}, \\ \varphi_1 \le \varphi_2 \Longrightarrow \int_I \varphi_1 &\le \int_I \varphi_2. \end{aligned}$$

PROOF. The second property is completely elementary. For the others we take partitions for φ_1 and φ_2, respectively, and form the intersections of the rectangles to produce a new partition that works for both φ_1 and φ_2. Thus we may write $I = \bigcup_{j=1}^N I_j$ and

$$\begin{aligned} \varphi_1 &= c_{1j} \quad \text{on the interior of } I_j, \\ \varphi_2 &= c_{2j} \quad \text{on the interior of } I_j. \end{aligned}$$

Then

$$\varphi_1 + \varphi_2 = c_{1j} + c_{2j} \quad \text{on the interior of } I_j$$

and this produces the equation for $\int_I (\varphi_1 + \varphi_2)$. For the inequality we note that $\varphi_2 - \varphi_1 \ge 0$ and thus

$$\int_I (\varphi_2 - \varphi_1) \ge 0.$$

The addition property now gives

$$\begin{aligned}\int_I \varphi_2 &= \int_I \varphi_1 + \int_I (\varphi_2 - \varphi_1)\\ &\geq \int_I \varphi_1.\end{aligned}$$

QED

C. **The Riemann integral**

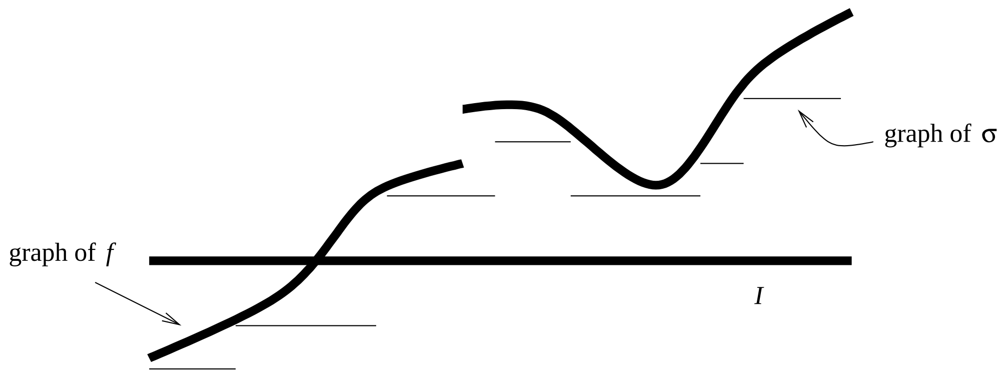
We now consider a fixed function

$$I \xrightarrow{f} \mathbb{R},$$

where I is a fixed special rectangle. We assume throughout that f is *bounded*; this means that there is a constant C such that

$$|f(x)| \leq C \qquad \text{for all } \; x \in I.$$

We now consider all possible step functions σ on I which are below f in the sense that $\sigma \leq f$. Here's a rendition of what the graphs of σ and f may resemble:



Since σ is a step function we can form the sum which is its integral. We certainly want all the resulting numbers to be less than or equal to the desired integral of f, and we hope that we can use these numbers to approximate what we want to call the integral of f. Thus we define with Darboux

$$\begin{aligned}&\text{the } \textit{lower integral} \text{ of } f \text{ over } I\\ &= \underline{\int_I} f\\ &= \sup \left\{ \int_I \sigma \;\middle|\; \sigma \leq f \right\}.\end{aligned}$$

We call this the *lower* integral because we do not yet know that it will serve as a useful definition of the integral.

Likewise we define using step functions $\tau \geq f$,

$$\begin{aligned}&\text{the \textit{upper integral} of } f \text{ over } I\\ &= \overline{\int_I} f\\ &= \inf\left\{ \int_I \tau \;\middle|\; \tau \geq f \right\}.\end{aligned}$$

Since $\sigma \leq f \leq \tau$ implies $\sigma \leq \tau$, we see that every lower sum $\int_I \sigma$ is less than or equal to every upper sum $\int_I \tau$. Thus

$$\underline{\int_I} f \leq \overline{\int_I} f.$$

PROBLEM 9–2. Show that if f is itself a step function,

$$\overline{\int_I} f = \underline{\int_I} f = \int_I f.$$

PROBLEM 9–3. If f and g are bounded functions on I, prove that

$$\overline{\int_I}(f+g) \leq \overline{\int_I} f + \overline{\int_I} g;$$
$$\underline{\int_I}(f+g) \geq \underline{\int_I} f + \underline{\int_I} g.$$

PROBLEM 9–4. Prove that

$$\underline{\int_I}(-f) = -\overline{\int_I} f.$$

PROBLEM 9–5. If $f \le g$ are bounded functions on I, prove that

$$\overline{\int_I} f \le \overline{\int_I} g;$$
$$\underline{\int_I} f \le \underline{\int_I} g.$$

The next problem gives "the" standard example of a function for which the lower and upper integrals are different.

PROBLEM 9–6. Suppose f is defined by

$$f(x) = \begin{cases} 0 & \text{if all coordinates of } x \text{ are rational,} \\ 1 & \text{otherwise.} \end{cases}$$

Prove that

$$\overline{\int_I} f = \operatorname{vol}(I);$$
$$\underline{\int_I} f = 0.$$

PROBLEM 9–7. Give examples of bounded functions f and g for which

$$\overline{\int_I}(f+g) < \overline{\int_I} f + \overline{\int_I} g.$$

Now we are ready to present the crucial

DEFINITION. If $I \xrightarrow{f} \mathbb{R}$ is bounded, then f is said to be *Riemann integrable* if

$$\underline{\int_I} f = \overline{\int_I} f.$$

In case f is integrable, we define its *Riemann integral* to be the common value of its lower and upper integrals, and we denote it as

$$\int_I f = \underline{\int_I} f = \overline{\int_I} f.$$

Notice that Problem 9–6 shows that some functions fail to be integrable. Also notice that Problem 9–2 shows that every step function is integrable.

There are other ways to define integrable functions, the most prominent being *Lebesgue* integrable. However, in this course we are going to deal with the definition given here, so we dispense with the adjective "Riemann" and just refer to integrable functions and the integral of a function.

PROBLEM 9–8. Suppose f and g are integrable on I and c is a constant. Prove that $f+g$ and cf are also integrable, and that

$$\begin{aligned} \int_I (f+g) &= \int_I f + \int_I g; \\ \int_I cf &= c\int_I f. \end{aligned}$$

Next we present an extremely important equivalent characterization of integrability. This will tell us a great deal about the role of supremum and infimum in our original definition.

CRITERION FOR INTEGRABILITY. *Assume* $I \xrightarrow{f} \mathbb{R}$ *is a bounded function. Then* f *is integrable* $\Longleftrightarrow$ *for every* $\epsilon > 0$ *there exist step functions* σ *and* τ *satisfying*

$$\begin{gathered} \sigma \le f \le \tau \quad \text{on } I, \\ \int_I \tau - \int_I \sigma < \epsilon. \end{gathered}$$

PROOF. Let us denote the lower and upper integrals of f by

$$\begin{aligned} \alpha &= \underline{\int}_I f, \\ \beta &= \overline{\int}_I f. \end{aligned}$$

If there exist step functions σ and τ as specified in the theorem, then by definition of the lower and upper integrals, respectively,

$$\begin{aligned} \alpha &\ge \int_I \sigma, \\ \beta &\le \int_I \tau. \end{aligned}$$

Therefore,

$$0 \le \beta - \alpha \le \int_I \tau - \int_I \sigma < \epsilon.$$

Therefore, for all $\epsilon > 0$ the number $\beta - \alpha$ satisfies the inequality

$$0 \le \beta - \alpha < \epsilon.$$

Of course, $\beta - \alpha$ is independent of ϵ. Therefore, as ϵ is arbitrary, we conclude that $\beta - \alpha = 0$, proving that f is integrable.

Conversely, suppose f is integrable. Then for any $\epsilon > 0$ the number $\alpha - \frac{\epsilon}{2}$, being smaller than α, is *not* an upper bound for the lower sums for f; i.e., there exists a step function $\sigma \le f$ such that

$$\int_I \sigma > \alpha - \frac{\epsilon}{2}.$$

Likewise, there exists $f \le \tau$ such that

$$\int_I \tau < \beta + \frac{\epsilon}{2}.$$

Since f is integrable we have $\alpha = \beta$, and subtraction gives

$$\int_I \tau - \int_I \sigma < \left(\beta + \frac{\epsilon}{2}\right) - \left(\alpha - \frac{\epsilon}{2}\right) = \epsilon.$$

QED

The importance of this criterion is that it does not require or even mention any knowledge of the lower or upper integrals for f. Consequently, we may detect integrability of f without having to know the numerical value of the integral itself. Here is an example:

THEOREM. *If f and g are integrable on I, then so is fg.*

PROOF. **Case 1**: f and g are both nonnegative functions. Since these functions are bounded, there exists a positive constant C such that

$$0 \le f \le C \quad \text{and} \quad 0 \le g \le C \quad \text{on} \quad I.$$

In our discussion of f we may as well use only step functions σ and τ satisfying

$$0 \le \sigma \le f \le \tau \le C,$$

since any value of σ which is less than 0 may be replaced with 0, and any value of τ which is greater than C may be replaced with C. Likewise for g. Thus for any $\epsilon > 0$, the integrability of f and g imply that step functions exist which satisfy

$$\begin{gathered} 0 \le \sigma_1 \le f \le \tau_1 \le C, \\ 0 \le \sigma_2 \le g \le \tau_2 \le C, \\ \int_I \tau_1 - \int_I \sigma_1 < \epsilon/2C, \\ \int_I \tau_2 - \int_I \sigma_2 < \epsilon/2C. \end{gathered}$$

Of course, we have used the integrability criterion here.

Then we have the inequalities

$$\sigma_1\sigma_2 \le fg \le \tau_1\tau_2,$$

and

$$\begin{aligned}\tau_1\tau_2 - \sigma_1\sigma_2 &= (\tau_1-\sigma_1)\tau_2 + \sigma_1(\tau_2-\sigma_2)\\ &\le C(\tau_1-\sigma_1) + C(\tau_2-\sigma_2),\end{aligned}$$

so that

$$\begin{aligned}\int_I \tau_1\tau_2 - \int_I \sigma_1\sigma_2 &\le C\int_I(\tau_1-\sigma_1) + C\int_I(\tau_2-\sigma_2)\\ &< \epsilon/2+\epsilon/2\\ &= \epsilon.\end{aligned}$$

By the integrability criterion again we conclude that fg is integrable.

Case 2: We now assume that f and g are integrable. As these functions are bounded, there exists a positive constant C such that

$$|f| \le C \quad \text{and} \quad |g| \le C \quad \text{on } I.$$

In particular, $f+C \ge 0$ and $g+C \ge 0$. Thanks to Problem 9–7, these functions $f+C$ and $g+C$ are integrable. We conclude from Case 1 that their product $(f+C)(g+C)$ is integrable. Thus

$$fg = (f+C)(g+C) - Cf - Cg - C^2$$

is also integrable, being a sum of four integrable functions.

QED

Don't miss the logic of what we've accomplished here. We have no hope at all of *calculating* the integral of fg. This is very different from the situation of Problem 9–8, where you were able to prove $f+g$ is integrable quite directly, for there was a good candidate for the value of its integral, namely the integral of f plus the integral of g.

There is one more item to handle before we move to the next section, namely the absolute value of a function. We first consider the following concept for real numbers:

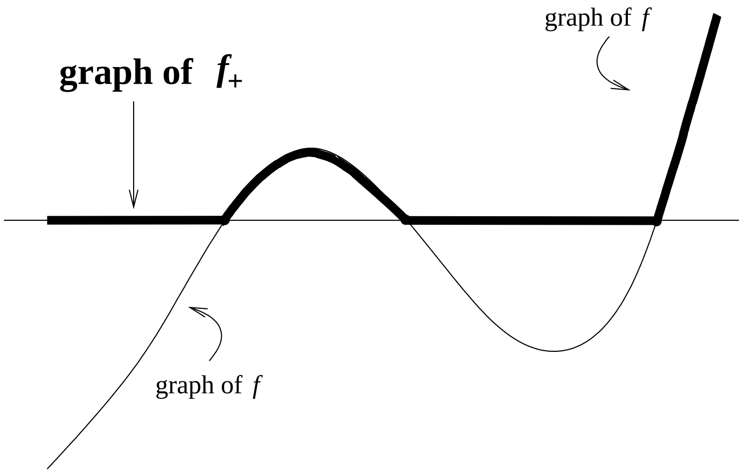
the *positive part* of any $t \in \mathbb{R}$ is the number

$$t_+ = \begin{cases} t & \text{if } t \ge 0,\\ 0 & \text{if } t \le 0;\end{cases}$$

the positive part of a real-valued function f is the function f_+ defined by

$$f_+(x) = (f(x))_+ \qquad \text{for all } x.$$

Typical sketch:



PROBLEM 9–9. Prove that if $I \xrightarrow{f} \mathbb{R}$ is integrable, then so is f_+.
(HINT: use the criterion for integrability, starting with $\sigma \le f \le \tau$. Show that $\sigma_+ \le f_+ \le \tau_+$. Compare $\tau_+ - \sigma_+$ to $\tau - \sigma$.)

THEOREM. *Suppose $I \xrightarrow{f} \mathbb{R}$ is integrable. Then $|f|$ is also integrable, and*

$$\left| \int_I f \right| \le \int_I |f|.$$

PROOF. It is easy to see that $|t| = t_+ + (-t)_+$. Thus $|f| = f_+ + (-f)_+$ is integrable, thanks to the preceding problem. Moreover, $f \le |f|$ implies

$$\int_I f \le \int_I |f|.$$

Likewise,

$$\int_I -|f| \le \int_I f.$$

QED

PROBLEM 9–10. Give an example of a nonintegrable function f whose absolute value $|f|$ is integrable.

PROBLEM 9–11. Suppose that f is integrable on I and that for all $x \in I$, $f(x) \ne 0$. Assume that $1/f$ is bounded. Prove that $1/f$ is integrable.

As we have seen in Problems 9–3 and 9–7, the upper integral does not behave linearly with respect to addition of functions. However, there are some useful results in cases in which one of the functions is integrable:

THEOREM. *Let f and g be bounded functions defined on the special rectangle $I \subset \mathbb{R}^n$. Assume f is integrable. Then*

$$\overline{\int_I}(f+g) = \int_I f + \overline{\int_I} g.$$

PROOF. By Problem 9–3 we have the inequality

$$\overline{\int_I}(f+g) \le \int_I f + \overline{\int_I} g.$$

Now apply Problem 9–3 again to achieve

$$\begin{aligned}\overline{\int_I} g &= \overline{\int_I}[f+g+(-f)] \\ &\le \overline{\int_I}(f+g) + \int_I(-f) \\ &= \overline{\int_I}(f+g) - \int_I f.\end{aligned}$$

This is the reverse inequality which we require.

QED

This theorem even has a nice converse:

THEOREM. *Let f be a bounded function on I. Assume that*

$$\overline{\int_I}(f+g) = \overline{\int_I} f + \overline{\int_I} g$$

for all bounded functions g. Then f is integrable.

PROOF. Too easy! Just set $g = -f$ and use Problem 9–4.

QED

PROBLEM 9–12. Prove that the bounded function f is integrable $\Longleftrightarrow$

$$\overline{\int_I} f = \overline{\int_I}(f-g) + \underline{\int_I} g$$

for all bounded functions g.

D. **Sufficient conditions for integrability**

Everything we have accomplished to this point is elegant and interesting, but there is one issue that needs treating. Namely, are there interesting functions which are integrable? Essentially all we know at the present time is that step functions are integrable. However, all is well. Most functions that arise in calculus are indeed integrable. We now present two situations. The first is in the context of single-variable calculus, and is quite interesting for its usefulness and the elegance of the proof.

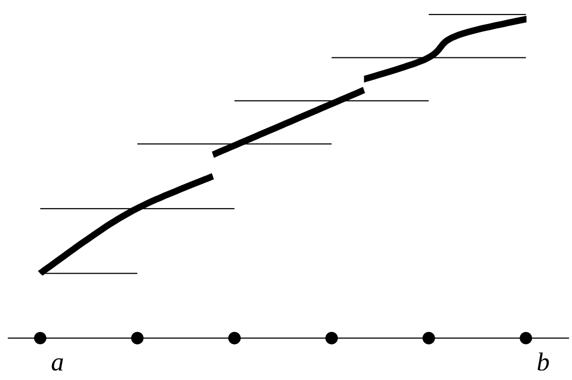
THEOREM. *Let* $[a, b] \xrightarrow{f} \mathbb{R}$ *be an increasing function. Then* f *is integrable.*

PROOF. The hypothesis means that $s \le t \Rightarrow f(s) \le f(t)$. In particular, $f(a) \le f(t) \le f(b)$, so f is bounded. Consider any partition of $[a, b]$, and denote the end points of the constituent intervals as

$$a = t_0 < t_1 < \cdots < t_m = b.$$

We then define step functions σ and τ by

$$\begin{aligned} \sigma(x) &= f(t_{i-1}) \quad \text{for } t_{i-1} \le x < t_i, \\ \tau(x) &= f(t_i) \quad \text{for } t_{i-1} \le x < t_i. \end{aligned}$$



The increasing nature of f guarantees that $\sigma \le f \le \tau$. Moreover,

$$\begin{aligned} \int_{[a,b]} \tau - \int_{[a,b]} \sigma &= \int_{[a,b]} (\tau - \sigma) \\ &= \sum_{i=1}^{m} (f(t_i) - f(t_{i-1}))(t_i - t_{i-1}). \end{aligned}$$

For any given $\epsilon > 0$, we choose the partition so that

$$t_i - t_{i-1} \le \frac{\epsilon}{f(b) - f(a)} \qquad \text{for } 1 \le i \le m.$$

Then

$$\int_{[a,b]} \tau - \int_{[a,b]} \sigma \;\le\; \sum_{i=1}^{m} (f(t_i) - f(t_{i-1})) \frac{\epsilon}{f(b) - f(a)}$$
$$= \epsilon.$$

This shows that f satisfies the criterion for integrability.

QED

Obviously, decreasing functions are also integrable.

The other situation is quite general for $\mathbb{R}^n$, and it is that f is *continuous*. Our proof shall actually use the stronger property, that f is *uniformly* continuous. This is not an actual restriction, since f is defined on the compact set I and a theorem of basic analysis asserts that every continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous.

THEOREM. *Let $I \xrightarrow{f} \mathbb{R}$ be continuous. Then f is integrable.*

PROOF. Since f is uniformly continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x - y\| < \delta$, then $|f(x) - f(y)| < \epsilon$. We then partition I into small enough rectangles $I_1, \dots, I_N$ to insure that if $x, y \in I_j$ for some j, then $\|x - y\| < \delta$ and thus $|f(x) - f(y)| < \epsilon$. Let p_j be the center of I_j and define step functions as follows:

$$\sigma(x) \;=\; f(p_j) - \epsilon \qquad \text{for } x \in \text{interior of } I_j,$$
$$\tau(x) \;=\; f(p_j) + \epsilon \qquad \text{for } x \in \text{interior of } I_j.$$

(Technical point: as f is bounded, say $|f| \le C$, we may simply choose σ to equal $-C$ and τ to equal C on the boundaries of the I_j's.) Then $\sigma \le f \le \tau$ on I. The proof goes like this: if $x \in$ interior of I_j, then $|f(x) - f(p_j)| < \epsilon$ and thus

$$\sigma(x) \;=\; f(p_j) - \epsilon \;<\; f(x).$$

Now we simply calculate

$$\int_I \tau - \int_I \sigma \;=\; \int_I (\tau - \sigma)$$
$$= \sum_{i=1}^{N} 2\epsilon \operatorname{vol}(I_j)$$
$$= 2\epsilon \operatorname{vol}(I).$$

As $\epsilon > 0$ is arbitrary, f satisfies the integrability criterion.

QED

E. **The fundamental theorem**

Since you are reading this material on n-dimensional calculus, it is presumed that you are already familiar with the great and wonderful fundamental theorem of calculus. Nevertheless, this is a logical place to present it, and the proof is short. The context is real-valued functions on $\mathbb{R}$. For a given interval $I = [a, b]$, we write

$$\int_I f = \int_a^b f,$$
$$\overline{\int_I} f = \overline{\int_a^b} f, \quad \text{etc.}$$

LEMMA. *Let $a < b < c$ and let $[a, c] \xrightarrow{f} \mathbb{R}$ be a bounded function. Then*

1. $\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f$, *likewise for lower integrals.*
2. f *is integrable over* $[a, c] \iff f$ *is integrable over* $[a, b]$ *and over* $[b, c]$.
3. *In case* f *is integrable over* $[a, c]$,

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

PROOF. There is a natural correspondence between step functions τ on $[a, c]$ which satisfy $f \le \tau$, and *pairs* of step functions τ_1 on $[a, b]$ and τ_2 on $[b, c]$ which satisfy $f \le \tau_1$ on $[a, b]$ and $f \le \tau_2$ on $[b, c]$. Namely, τ_1 and τ_2 may be taken to be the restrictions of τ to the respective intervals; and on the other hand for given τ_1 and τ_2 we may define

$$\tau = \begin{cases} \tau_1 & \text{on } [a, b), \\ \tau_2 & \text{on } (b, c], \end{cases}$$

and simply let $\tau(b)$ be any value greater than $f(b)$.

In the above correspondence we have

$$\int_a^c \tau = \int_a^b \tau_1 + \int_b^c \tau_2.$$

From this the first statement follows easily for upper integrals. The corresponding result for lower integrals is entirely similar.

We then easily conclude that the $\Leftarrow$ portion of the second statement is valid. Conversely, suppose f is integrable on $[a, c]$. Then from the first statement,

$$\overline{\int_a^b} f + \overline{\int_b^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f.$$

As each term on the left side is at least as large as the corresponding term on the right side, we conclude that

$$\overline{\int_a^b} f = \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_b^c} f = \underline{\int_b^c} f.$$

Finally, the third statement is now immediate.

QED

FUNDAMENTAL THEOREM OF CALCULUS. *Assume that* $[a,b] \xrightarrow{f} \mathbb{R}$ *is integrable. Then* f *is integrable over each subinterval* $[a,x]$, *so it is possible to define the function*

$$F(x) \;=\; \int_a^x f.$$

Assume that f *is continuous at* x_0. *Then* F *is differentiable at* x_0, *and*

$$F'(x_0) \;=\; f(x_0).$$

PROOF. Consider arbitrary s, t such that $a \le s \le x_0 \le t \le b$, and $s < t$. Then

$$\begin{aligned}
\frac{F(t)-F(s)}{t-s} &= \frac{1}{t-s}\left(\int_a^t f - \int_a^s f\right) \\
&= \frac{1}{t-s}\int_s^t f \\
&= \frac{1}{t-s}\int_s^t (f(x_0) + f - f(x_0)) \\
&= \frac{1}{t-s}\left(f(x_0)(t-s) + \int_s^t (f - f(x_0))\right) \\
&= f(x_0) + \frac{1}{t-s}\int_s^t (f - f(x_0)).
\end{aligned}$$

For any $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| \le \delta$, then $|f(x) - f(x_0)| \le \epsilon$. Therefore if $0 < t - s \le \delta$,

$$\begin{aligned}
\left|\frac{F(t)-F(s)}{t-s} - f(x_0)\right| &= \left|\frac{1}{t-s}\int_s^t (f - f(x_0))\right| \\
&\le \frac{1}{t-s}\int_s^t |f - f(x_0)| \\
&\le \frac{1}{t-s}\int_s^t \epsilon \\
&= \epsilon.
\end{aligned}$$

We conclude that $F'(x_0)$ exists and equals $f(x_0)$.

QED

COROLLARY. *Let* $[a,b] \xrightarrow{g} \mathbb{R}$ *be of class* C^1. *Then*

$$\int_a^b g' \;=\; g(b) - g(a).$$

PROOF. Define

$$H(x) \;=\; \int_a^x g' - g(x).$$

(Since g' is continuous on $[a,b]$, it is integrable.) By the FTC,

$$H'(x) = g'(x) - g'(x) = 0,$$

for all $a \le x \le b$. Therefore the *mean value theorem* implies that H is a constant function. In particular, $H(b) = H(a)$; that is,

$$\int_a^b g' - g(b) = -g(a).$$

QED

PROBLEM 9–13. Show that the proof of FTC actually proves the following generalization: suppose only that $[a,b] \xrightarrow{f} \mathbb{R}$ is bounded, and that f is continuous at x_0. Define

$$F(x) = \overline{\int_a^x} f.$$

Then $F'(x_0) = f(x_0)$.

PROBLEM 9–14. Here's a different proof (for $n = 1$) that continuity implies integrability. Assume that $[a,b] \xrightarrow{f} \mathbb{R}$ is continuous. Then define

$$F(x) = \overline{\int_a^x} f, \quad G(x) = \underline{\int_a^x} f$$

(we're not using the theorem that continuity $\Rightarrow$ integrability). Using Problem 9–13, prove that $F(x) = G(x)$.

One feature of this problem is that the uniform continuity of f is not required for the conclusion. Not a particularly striking advantage, knowing that f *is* uniformly continuous!

F. **The Fubini theorem**

At the present time we are aware of just one major result for the actual computation of integrals, the fundamental theorem of calculus. Beautiful and useful as it is, it applies only to single variable integration. In the present section we are going to present a wonderful theorem which theoretically reduces all computation of n-dimensional integrals to 1-dimensional ones. This result, the *Fubini* theorem, thus places us in a position to use the FTC for our computations. Therefore in the section after this one we shall quickly compute a great variety of integrals.

1. **The notational set-up**

It becomes helpful at this juncture to introduce some extra notation for our integrals. This will be the familiar "dummy variable" notation from basic calculus. Thus if we are integrating over the special rectangle $I \subset \mathbb{R}^n$, we can use any of the following notations:

$$\int_I f,$$
$$\int_I f \; d\text{vol},$$
$$\int_I f(x)dx,$$
$$\int_I f(x_1, \ldots, x_n)dx_1 \; \ldots \; dx_n,$$
$$\int_I f(y)dy,$$
$$\int_I f(u)du.$$

It's the third of these that is so helpful in the present section.

Now let ℓ and m be positive integers, and $n = \ell + m$. We think of $\mathbb{R}^n$ as a Cartesian product,

$$\mathbb{R}^n \; = \; \mathbb{R}^\ell \times \mathbb{R}^m,$$

and we strive to designate points in $\mathbb{R}^n$ consistently in the following manner:

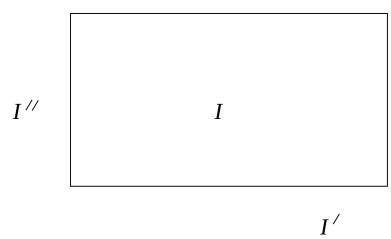
$$z \in \mathbb{R}^n; \;\; x \in \mathbb{R}^\ell, \;\; y \in \mathbb{R}^m,$$
$$z \; = \; (x, y).$$

Of course, we mean by this notation that

$$z_i \; = \; x_i \qquad \text{for} \;\; 1 \le i \le \ell,$$
$$z_i \; = \; y_{i-\ell} \qquad \text{for} \;\; \ell + 1 \le i \le n.$$

We also write with obvious notation our special rectangle I as the Cartesian product of two special rectangles:

$$I \; = \; I' \times I''.$$



Here is the Fubini formula we are going to be discussing and proving:

$$\int_I f(z)dz = \int_{I'} \left(\int_{I''} f(x,y)dy \right) dx. \qquad \text{(Fubini)}$$

The notation is supposed to mean this: for each $x \in I'$ consider the function $f(x,\cdot)$ on I''. Integrate this function over I'' to obtain a number depending on x, say

$$F(x) = \int_{I''} f(x,y)dy.$$

Then (Fubini) is the assertion that

$$\int_I f(z)dz = \int_{I'} F(x)dx.$$

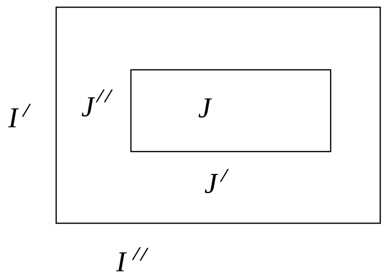
Of course, we certainly must take care in the matter of the integrability of the various functions in this formula.

2. **The case of step functions**

Here we show that (Fubini) is valid for any step function defined on I. Observe that if (Fubini) is valid for certain functions $f_1, \dots, f_N$, then it is also valid for linear combinations of the form $f = c_1 f_1 + \cdots + c_N f_N$, where each c_j is a constant. Therefore it will suffice here to prove (Fubini) for the simplest possible type of step function. Namely, for a given special rectangle $J \subset I$, consider the function

$$f(z) = \begin{cases} 1 & \text{if } z \in J, \\ 0 & \text{if } z \in I - J. \end{cases}$$

The proof of (Fubini) is then the following elementary verification. Write as a Cartesian product $J = J' \times J''$. Then if $x \in J'$ the function $f(x,\cdot)$ equals 1 on J'', 0 elsewhere. Thus $f(x,\cdot)$ is a step function on I'' and its integral $F(x) = \mathrm{vol}_m(J'')$ in this case. On the other hand, if $x \in I' - J'$, then $f(x,\cdot) = 0$, so $F(x) = 0$. Thus we have shown that



$$F(x) = \begin{cases} \mathrm{vol}_m(J'') & \text{if } x \in J', \\ 0 & \text{if } x \in I' - J'. \end{cases}$$

We see that F is a step function on I', and we calculate

$$\begin{aligned} \int_{I'} F(x)dx &= \mathrm{vol}_m(J'')\mathrm{vol}_\ell(J') \\ &= \mathrm{vol}_n(J' \times J'') \\ &= \mathrm{vol}_n(J) \\ &= \int_I f(z)dz. \end{aligned}$$

This concludes the present section. Note of course that the original formula for volume of rectangles as the product of edge lengths is what makes this work.

PROBLEM 9–14. Using the above notation, suppose that g is an integrable function on I' and h is an integrable function on I''. Prove that their "tensor product"

$$f(x, y) = g(x)h(y)$$

is integrable on $I = I' \times I''$ and that

$$\int_I f(z)dz = \int_{I'} g(x)dx \int_{I''} h(y)dy.$$

3. **The case of upper and lower integrals**

We now assume that $I \xrightarrow{f} \mathbb{R}$ is any bounded function. We keep the notation from the preceding sections. We try to analyze the upper integral of f, so we consider any step function $\tau \geq f$. Then for any fixed $x \in I'$ define

$$T(x) \;=\; \int_{I''} \tau(x, y)dy.$$

Since $\tau \geq f$, we have $\tau(x, \cdot) \geq f(x, \cdot)$, so we conclude that

$$T(x) \;\geq\; \overline{\int_{I''}} f(x, y)dy.$$

Let us define $F(x)$ to be the integral on the right side:

$$F(x) = \overline{\int_{I''}} f(x, y)dy.$$

(Notice that $F(x)$ has to be the upper integral, as we have no hypothesis guaranteeing $f(x, \cdot)$ to be integrable.) Our inequality,

$$T(x) \;\geq\; F(x) \qquad \text{for all } \; x \in I',$$

asserts that the step function T is a competitor for the calculation of the upper integral of F, so we conclude that

$$\int_{I'} T(x)dx \;\geq\; \overline{\int_{I'}} F(x)dx.$$

By the relation (Fubini) for τ, the last inequality can be written

$$\int_I \tau dz \;\geq\; \overline{\int_{I'}} F(x)dx.$$

Finally, the infimum of the set of all such $\int_I \tau$ is the upper integral of f, so we have proved

$$\overline{\int_I} f dz \ \geq \ \overline{\int_{I'}} F(x)dx.$$

A corresponding inequality holds for lower integrals as well. Thus we have proved the following

LEMMA. *Let $I \xrightarrow{f} \mathbb{R}$ be a bounded function. Then*

$$\begin{aligned} \underline{\int_I} f(z)dz \ &\leq \ \underline{\int_{I'}} \left(\underline{\int_{I''}} f(x,y)dy \right) dx \\ &\leq \ \overline{\int_{I'}} \left(\overline{\int_{I''}} f(x,y)dy \right) dx \\ &\leq \ \overline{\int_I} f(z)dz. \end{aligned}$$

4. **Statement and proof**

FUBINI'S THEOREM. *Following the above notation, assume that $I \xrightarrow{f} \mathbb{R}$ is integrable. Define*

$$\begin{aligned} F_1(x) \ &= \ \underline{\int_{I''}} f(x,y)dy, \\ F_2(x) \ &= \ \overline{\int_{I''}} f(x,y)dy. \end{aligned}$$

Then of course $F_1 \leq F_2$ on I'. Both F_1 and F_2 are integrable over I, and

$$\int_I f(z)dz \ = \ \int_{I'} F_1(x)dx \ = \ \int_{I'} F_2(x)dx.$$

PROOF. We have from Section 3

$$\begin{aligned} \int_I f(z)dz \ &\leq \ \underline{\int_{I'}} F_1(x)dx \\ &\leq \ \underline{\int_{I'}} F_2(x)dx \\ &\leq \ \overline{\int_{I'}} F_2(x)dx \\ &\leq \ \int_I f(z)dz. \end{aligned}$$

Because the smallest and largest numbers in this string of inequalities are the same, these must all be equalities. Therefore the upper and lower integrals of F_2 are equal, proving F_2 is integrable, and also proving at the same time the formula for its integral. The same sort of proof holds for F_1.

QED

COROLLARY. *Assume* $I \xrightarrow{f} \mathbb{R}$ *is integrable and also that for each* $x \in I$, $f(x,\cdot)$ *is integrable over* I''. *Then*

$$\int_I f(z)dz = \int_{I'} \left(\int_{I''} f(x,y)dy \right) dx.$$

PROOF. With the extra hypothesis we have $F_1(x) = F_2(x)$.

QED

G. **Iterated integrals**

The expression on the right side of the equation we have just found is called an *iterated* integral. It is not an actual n-dimensional integral as the left side is. Therein lies the power of the Fubini theorem, after all: computing an n-dimensional integral is often a matter of being able to compute a succession of 1-dimensional integrals.

EXAMPLE. Let $I = [0,1] \times [0,1]$ in $\mathbb{R}^2$, and consider the integral

$$\int_I ye^{xy}.$$

The integrand is of course continuous, so this fits Fubini's theorem. We could choose to perform either an x-integration or a y-integration first. It appears that the x-integration is easier, so we write the integral in iterated fashion as

$$\begin{aligned}
\int_0^1 \left(\int_0^1 ye^{xy}dx \right) dy &= \int_0^1 e^{xy} \Big|_{x=0}^{x=1} dy \\
&= \int_0^1 (e^y - 1)dy \\
&= (e^y - y) \Big|_{y=0}^{y=1} \\
&= e - 2.
\end{aligned}$$

Be sure to notice the nice interplay between Fubini's theorem and the FTC.

> **PROBLEM 9–16.** Evaluate the integral we have just computed by performing the y integration first.

Another useful application of Fubini's theorem is the *interchange of the order of integration* in an iterated integral:

$$\int_{I'} \left(\int_{I''} f(x,y)dy \right) dx = \int_{I''} \left(\int_{I'} f(x,y)dx \right) dy.$$

The validity of this equation is of course a direct consequence of the Fubini theorem, as both sides are equal to the integral of f over $I' \times I''$. (The proof of the Fubini theorem was oblivious to the order of doing things.) Incidentally, we often write these integrals without the parentheses, so that the left side above becomes

$$\int_{I'}\int_{I''} f(x,y)dydx.$$

This must be read properly, from the inside out: dy goes with I'', dx goes with I'. E.g., here are two quite different integrals:

$$\int_0^3\int_{-1}^1 xdxdy = 0,$$
$$\int_0^3\int_{-1}^1 xdydx = 9.$$

PROBLEM 9–17. Evaluate the integral

$$\int_0^3\int_0^2 x^3y^2\cos(x^2y^3)dxdy.$$

(Answer: (1-cos 108)/162)

One must beware of seeming contradictions in cases in which the integrand is *not* integrable. Try the next two problems as examples.

PROBLEM 9–18. Compute the iterated integral

$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dx\right)dy.$$

(Answer: $-1/2$) Then compute the reversed one,

$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dy\right)dx.$$

PROBLEM 9–19. Repeat Problem 9–18 but with the integrand

$$f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

(Answer: $\pm\pi/4$)

PROBLEM 9–20. In the two preceding problems how can you compute the second integral "by inspection," once you have computed the first?

PROBLEM 9–21. From the 1989 William Lowell Putnam Mathematical Competition, Problem A-2

$$\text{Evaluate } \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy dx \text{ where } a \text{ and } b \text{ are positive.}$$

(Answer: $(e^{a^2b^2} - 1)/ab$)

The "counterexamples" of Problems 9–17 and 18 are based upon the fact that f is not integrable. In each example the function has a severe discontinuity at the origin.

It is a remarkable fact that if the integrand does not change sign, then all is well. That is, if f is reasonably well behaved (e.g. continuous) and does not change sign, then the Fubini conclusion holds. The integrals may be "improper" because of unboundedness of the integrand or the region of integration, and in fact the improper integrals may diverge. In all such cases the Fubini conclusions are valid, even if both sides are infinite. This principle is very much like a conservation of mass situation, in which the integrand represents a density. Fubini's theorem then asserts that the total mass, even if it is infinite, is always the same. It is only when there are infinite amounts of both positive and negative masses that trouble can arise.

To prove the assertion of the above paragraph would take us too far from calculus. See a book on Lebesgue integration for a thorough discussion.

We now present an example of tremendous significance. We are going to use Fubini's theorem to compute the famous and important *Gaussian integral*

$$\int_0^\infty e^{-x^2} dx.$$

We first observe that this integral is finite. Although we have no technique for integrating e^{-x^2}, yet this function is smaller than $2xe^{-x^2}$ for $x > \frac{1}{2}$, so we have the comparison

$$\begin{aligned}\int_{1/2}^\infty e^{-x^2} dx &< \int_{1/2}^\infty 2xe^{-x^2} dx \\ &= -e^{-x^2} \Big|_{1/2}^\infty \\ &= e^{-1/4}.\end{aligned}$$

We are now going to apply Fubini's theorem to the function $ye^{-y^2(1+x^2)}$ integrated over the infinite region consisting of the entire first quadrant in the $x - y$ plane. We shall use our unproven observation of the preceding page, that since the integrand is positive and continuous, Fubini's theorem still holds. It would not be difficult to integrate over a finite rectangle $[0, a] \times [0, b]$ and then let a, $b \to \infty$, but that really seems to disguise the elegance of the entire argument.

Let us designate by A the number we are going to compute:

$$A = \int_0^\infty e^{-x^2} dx.$$

Then we have from Fubini's theorem

$$\begin{aligned}
\int_{\text{first quadrant}} ye^{-y^2(1+x^2)} &= \int_0^\infty \int_0^\infty ye^{-y^2(1+x^2)}dydx \\
&= \int_0^\infty -\frac{1}{2(1+x^2)}e^{-y^2(1+x^2)}\Big|_{y=0}^{y=\infty} dx \\
&= \int_0^\infty \frac{1}{2(1+x^2)}dx \\
&= \frac{1}{2}\arctan x \Big|_{x=0}^{x=\infty} \\
&= \frac{\pi}{4}.
\end{aligned}$$

On the other hand, if we perform the x-integration first, the above integral equals

$$\begin{aligned}
\int_0^\infty \int_0^\infty ye^{-y^2(1+x^2)}dxdy &= \int_0^\infty e^{-y^2} \int_0^\infty e^{-y^2x^2}ydxdy \\
&\overset{yx=t}{=} \int_0^\infty e^{-y^2} \int_0^\infty e^{-t^2}dtdy \\
&= \int_0^\infty e^{-y^2}Ady \\
&= A\int_0^\infty e^{-y^2}dy \\
&= A^2.
\end{aligned}$$

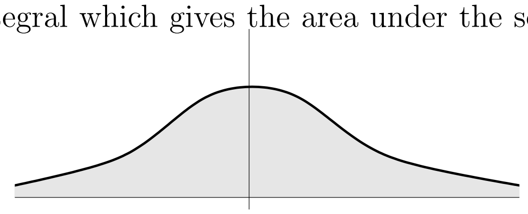
Thus $A^2 = \pi/4$. This gives the result we want:

$$\boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.}$$

This is a truly impressive accomplishment. Although there is no elementary function whose derivative is e^{-x^2}, and thus the indefinite integral of e^{-x^2} cannot be "calculated," yet the above infinite improper integral is known "in closed form"!

A companion of the formula is the integral which gives the area under the so-called bell-shaped curve,

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}.$$



An analogous situation occurs in deriving the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx,$$

another example in which the indefinite integral cannot be calculated.

PROBLEM 9–22. Integrate the function $e^{-xy}\sin x$ over the rectangle $[0,a]\times[0,b]$. Obtain the formula

$$\int_0^a \frac{1-e^{-bx}}{x}\sin x dx \;=\; \arctan b - \int_0^b e^{-ay}\frac{\cos a + y\sin a}{y^2+1}dy.$$

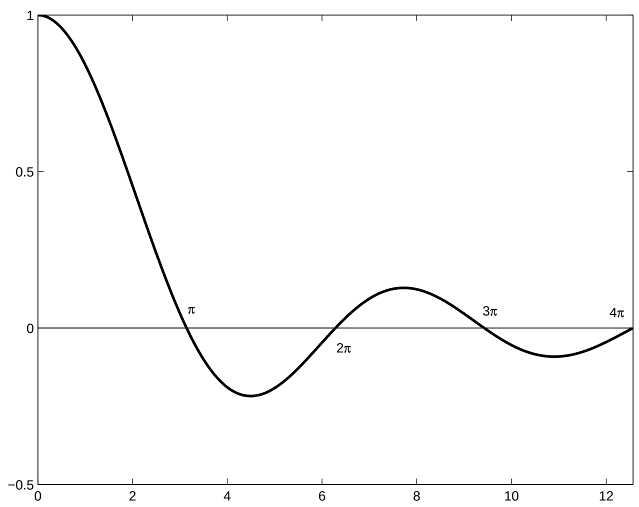
PROBLEM 9–23. Show that letting $b\to\infty$ in the above formula produces

$$\int_0^a \frac{\sin x}{x}dx \;=\; \frac{\pi}{2} - \int_0^\infty e^{-ay}\frac{\cos a + y\sin a}{y^2+1}dy.$$

Show that the integrand on the right side has absolute value less than e^{-ay} and then show that

$$\lim_{a\to\infty}\int_0^a \frac{\sin x}{x}dx \;=\; \frac{\pi}{2}.$$

The graph of the function $\sin x/x$ has the rough appearance:



The *signed* area between the graph and the x-axis therefore has the value

$$\int_0^\infty \frac{\sin x}{x}dx \;=\; \frac{\pi}{2}.$$

PROBLEM 9–24. Show that the total area represented above is infinite. That is, show that

$$\int_0^\infty \frac{|\sin x|}{x} dx = \infty.$$

(HINT: estimate each integral $\displaystyle\int_{\pi j}^{\pi j+\pi} \frac{|\sin x|}{x} dx$ as greater than a constant times j^{-1}.)

PROBLEM 9–25. Define

$$f(x) = \int_0^x \frac{\sin t}{t} dt.$$

Of course, f is a continuous function for $0 \le x < \infty$ with limit $\pi/2$ as $x \to \infty$. Prove that the maximum value of f is $f(\pi)$.

The number $f(\pi)$ is sometimes called the *Gibbs constant*, and

$$f(\pi) = 1.85193705\ldots\ .$$

A related constant is also sometimes called the Gibbs constant:

$$\frac{2}{\pi} f(\pi) = 1.17897974\ldots\ .$$

These numbers are important in Fourier series discussions.

PROBLEM 9–26. Use an integration by parts to show that

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.$$

PROBLEM 9–27. From Problem 9–25 derive the result

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

PROBLEM 9–28. Show that

$$\int_0^\infty \left(\frac{\sin ax}{x}\right)^2 dx = \frac{\pi}{2}|a|.$$

PROBLEM 9–29. Show that for $0 \le a \le b$

$$\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi}{2} a.$$

PROBLEM 9–30*. Show that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8}.$$

H. **Volume**

Our goal in this section is to discuss the theoretical aspects of the n-dimensional volume of sets contained in $\mathbb{R}^n$. We postpone the art of calculating volumes of interesting sets to Chapter 10. The entire discussion is facilitated by regarding volume as a special case of integration. To accomplish this easily we introduce the

DEFINITION. Assume $A \subset \mathbb{R}^n$. The *indicator function* of A is the function 1_A:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbb{R}^n - A. \end{cases}$$

We would like to integrate 1_A and define the resulting number to be the volume of A. We shall assume A is bounded, so that there exists a special rectangle I such that $A \subset I$. Then the integral we wish to use can be written

$$\int_I 1_A.$$

A minor issue arises in that I is certainly not unique. However, since 1_A is zero in $I - A$, the resulting integral will not depend on I anyway.

A major issue is the fact that 1_A might not be integrable. So we shall begin with suitably modified definitions.

DEFINITION. Let $A \subset \mathbb{R}^n$ be a bounded set. Let I be any special rectangle such that $A \subset I$. Then the *inner volume* of A is the number

$$\underline{\text{vol}}(A) = \underline{\int}_I 1_A,$$

and the *outer volume* of A is

$$\overline{\text{vol}}(A) = \overline{\int}_I 1_A.$$

Of course, $\underline{\mathrm{vol}}(A) \le \overline{\mathrm{vol}}(A)$. If these two numbers are equal, we say that A is *contented* and we write

$$\mathrm{vol}(A) \;=\; \textit{volume} \text{ of } A \;=\; \int_I 1_A.$$

The collection of contented sets in $\mathbb{R}^n$ has very pleasant features with respect to standard set operations. These follow at once:

THEOREM. *All special rectangles in $\mathbb{R}^n$ are contented and have volume as previously defined. If A and B are contented subsets of $\mathbb{R}^n$, then so are*

$$A \cap B, \quad A \cup B, \quad A - B.$$

Moreover,

$$\mathrm{vol}(A) + \mathrm{vol}(B) \;=\; \mathrm{vol}(A \cup B) + \mathrm{vol}(A \cap B).$$

PROOF. Since 1_A and 1_B are integrable, so is their product $1_A \cdot 1_B = 1_{A\cap B}$, thanks to the theorem on p. 9–10. This proves that $A \cap B$ is contented. Since $1_{A-B} = 1_A - 1_{A\cap B}$, the set $A - B$ is contented. Finally, since

$$1_A + 1_B \;=\; 1_{A\cup B} + 1_{A\cap B},$$

the remaining statements follow.

PROBLEM 9–31. Prove that if A, B, C are contented subsets of $\mathbb{R}^n$, then

$$\begin{aligned}\mathrm{vol}(A)+&\mathrm{vol}(B) + \mathrm{vol}(C) + \mathrm{vol}(A \cap B \cap C)\\ &= \mathrm{vol}(A \cup B \cup C) + \mathrm{vol}(A \cap B) + \mathrm{vol}(A \cap C) + \mathrm{vol}(B \cap C).\end{aligned}$$

PROBLEM 9–32. Since the empty set has $\mathrm{vol}(\emptyset) = 0$, the preceding problem implies the equation in the theorem. Extend the result of the preceding problem to the case of four contented sets. You should get eight volumes of sets on each side of your equation.

PROBLEM 9–33. A general result along these lines, involving contented sets $A_1, \dots, A_k$ in $\mathbb{R}^n$, is called the *inclusion-exclusion principle.* Express this principle in the form

$$\mathrm{vol}(A_1 \cup \cdots \cup A_k) = \mathrm{vol}(A_1) + \cdots + \mathrm{vol}(A_k) - \mathrm{vol}(A_1 \cap A_2) - \dots,$$

carefully writing out summations to include all the terms.

PROBLEM 9–34. Prove that if A and B are bounded sets, then

$$\overline{\text{vol}}(A \cup B) \ \le\ \overline{\text{vol}}(A) + \overline{\text{vol}}(B);$$

and if A and B are disjoint, then

$$\underline{\text{vol}}(A \cup B) \ \ge\ \underline{\text{vol}}(A) + \underline{\text{vol}}(B).$$

(Problems 9–36 and 9–37 strengthen these inequalities.)

PROBLEM 9–35. Prove that if $A \subset \mathbb{R}^n$ is a bounded set such that $\overline{\text{vol}}(A) = 0$, then A is contented.

PROBLEM 9–36. Prove that if $A \subset \mathbb{R}^n$ is a bounded countable set, then $\underline{\text{vol}}(A) = 0$. Give an example of such a set for which $\overline{\text{vol}}(A) > 0$.

The term *contented* is a type of historical accident. These concepts were introduced by Camille Jordan, and wherever we have written "volume" the phrase "Jordan content" is often used. Thus when the inner and outer content of a set are equal, the set is said to be contented.

Though we have used integration to define volume, it is an easy and important exercise to eliminate the integration symbol from the discussion. Thus suppose σ is a step function which enters into the competition for the definition of $\underline{\text{vol}}(A)$. That is, σ is a step function defined on I and

$$\sigma \le 1_A \quad \text{on} \ \ I.$$

Since we want the integral of σ to be as large as possible in this competition, any value of σ that is negative may certainly be replaced by 0; the new step function satisfies

$$0 \le \sigma \le 1_A \quad \text{on} \ \ I.$$

More importantly, suppose σ takes the constant value c_j on the interior of the rectangle $I_j \subset I$. If $0 < c_j < 1$, then the fact that $c_j \le 1_A$ on the interior of I_j implies $1_A > 0$ there. Thus 1_A takes the value 1 there. Thus we may increase c_j to 1 and the new step function is still $\le 1_A$.

The conclusion is this: in computing $\underline{\text{vol}}(A)$ we may employ only those step functions σ which take only the values 0 and 1 and which are $\le 1_A$. Corresponding to such a step function σ there is a partition of I into nonoverlapping special rectangles $I_1, \dots, I_N$. On the interior of each of these σ takes the value 0 or 1. If σ equals 1 on the interior of I_j, then 1_A equals 1 there as well. Thus, the interior of I_j is contained in A.

DEFINITION. A *special polygon* in $\mathbb{R}^n$ is any finite union of special rectangles.

We notice that if P is a special polygon, then P can be expressed as a finite union of nonoverlapping special rectangles,

$$P \ =\ \bigcup_{j=1}^{N} I_j,$$

and that

$$\mathrm{vol}(P) \;=\; \int_I 1_P \;=\; \sum_{j=1}^{N} \mathrm{vol}(I_j).$$

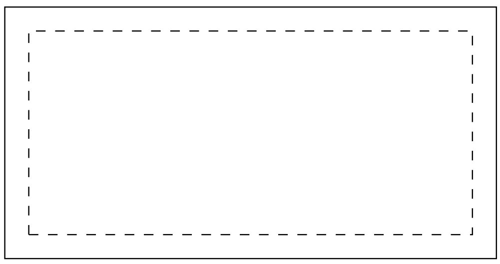
We therefore conclude that for any bounded set A,

$$\underline{\mathrm{vol}}(A) \;=\; \sup\{\mathrm{vol}(P) \mid P \;=\; \text{any special polygon} \;\; \subset A\}.$$

Analogously, we obtain the result that

$$\overline{\mathrm{vol}}(A) \;=\; \inf\{\mathrm{vol}(Q) \mid Q \;=\; \text{any special polygon} \;\; \supset A\}.$$

(There is a minor technicality we have skipped: the matter of whether $I_j \subset A$ or only the interior of $I_j \subset A$. Remember that our special rectangles are closed sets. This is no difficulty, since if only the interior of I_j is contained in A, then a slightly smaller special rectangle I_j' is contained in the interior of I_j; the loss in volume can be made as small as desired.



.)

These are the formulas we were hoping for. They display the inner and outer volumes of a set A entirely in terms of volumes of special polygons in a very intuitive way. The integration symbol is not required in these expressions.

PROBLEM 9–37. Prove that if A and B are bounded sets, then

$$\overline{\mathrm{vol}}(A \cup B) + \overline{\mathrm{vol}}(A \cap B) \;\le\; \overline{\mathrm{vol}}(A) + \overline{\mathrm{vol}}(B).$$

(HINT: start with $A \subset Q_1$ and $B \subset Q_2$.)

PROBLEM 9–38. Prove a similar inequality for inner volumes.

PROBLEM 9–39. Give an example of disjoint sets A and B such that

$$\overline{\mathrm{vol}}(A \cup B) < \overline{\mathrm{vol}}(A) + \overline{\mathrm{vol}}(B).$$

There is a beautiful balance between inner and outer volume, which we now consider.

THEOREM. *Let A be a bounded set and B a contented set. Then*

$$\text{vol}(B) \;=\; \underline{\text{vol}}(B \cap A) + \overline{\text{vol}}(B - A).$$

PROOF. This is an easy consequence of Problem 9–12, which implies that if f is integrable and g is bounded, then

$$\int_I f \;=\; \underline{\int}_I g + \overline{\int}_I (f - g).$$

Let I be a special rectangle containing B and choose the functions

$$f \;=\; 1_B, \quad g = 1_{B\cap A}.$$

Then it follows that

$$f - g \;=\; 1_{B-A}.$$

The result follows.

QED

PROBLEM 9–40. Let A be a bounded subset of $\mathbb{R}^n$. Prove that A is contented $\Longleftrightarrow$ for all bounded sets

$$\overline{\text{vol}}(B) \;=\; \overline{\text{vol}}(B \cap A) + \overline{\text{vol}}(B - A).$$

(HINT: for $\Leftarrow$ let B be a large rectangle; for $\Rightarrow$ start with $Q \supset B$ and prove $\geq$.)

PROBLEM 9–41. Prove a similar result using inner volumes instead.

Problem 9–40 is an extremely interesting result. It states that a set A is contented if and only if it enjoys the feature of "splitting every set just right according to outer volume." This means that if the set A is used to split an arbitrary bounded set B into a disjoint union

$$B \;=\; (B \cap A) \cup (B - A),$$

then in doing this the outer volumes of the two pieces add up to equal the outer volume of B. (Problem 9–41 gives the similar result for inner volumes.) This observation has an analog for the more advanced Lebesgue theory of integration (and volume), and has been used extensively in the fundamental theory of integration. Its utility stems from the fact that, in our case, a set can be detected to be contented by understanding outer volume only; inner volume does not even need to be introduced at all.

One additional issue of the utmost importance is going to be deferred to Chapter 10. Namely, for a *parallelogram* $P \subset \mathbb{R}^n$ we have already defined its volume in Chapter 8, using linear algebra, the Gram determinant, etc. In the present chapter we have defined quite independently $\underline{\text{vol}}(P)$ and $\overline{\text{vol}}(P)$. We certainly expect that P is contented and that its volume defined as $\underline{\text{vol}}(P) = \overline{\text{vol}}(P)$ is the same number as its volume as defined in Chapter 8. This is all true and not really difficult, but the efficient way to handle this issue is to develop the change of variables formula of Section 10F.

I. **Integration and volume**

Throughout this section we are going to be examining a bounded function $I \xrightarrow{f} \mathbb{R}$, where I is a special rectangle in $\mathbb{R}^n$. We shall also be assuming that $f \geq 0$. In this situation in classical calculus of one variable, the integral of f is identified as the area "under its graph," that is, the area between its graph and the "x-axis." It is this situation that we want to generalize and prove.

First we define our terms by introducing notation for the graph of a function, a concept we have already discussed in the explicit description of manifolds in Section 5D.

DEFINITION. The *graph* of $I \xrightarrow{f} \mathbb{R}$ is the subset of $\mathbb{R}^{n+1}$

$$\text{Graph}(f) = \{(x, f(x)) \mid x \in I\}.$$

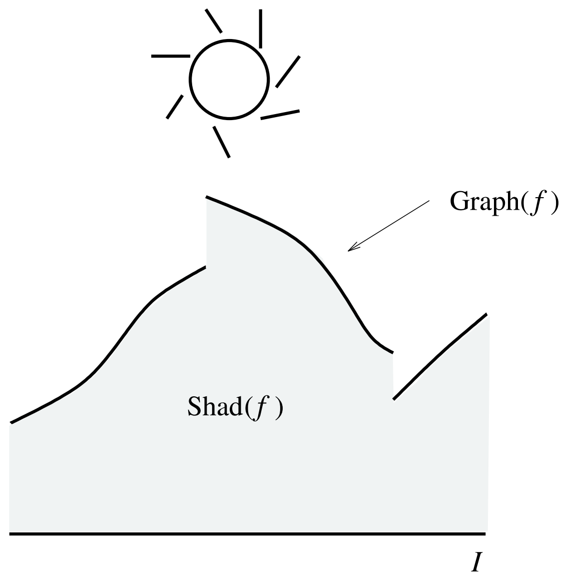
In other words, the points in Graph(f) are precisely the points of the form $(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$.

The above definition is quite general, and certainly does not require our assumption that $f \geq 0$. But the next definition definitely makes sense only in that setting.

DEFINITION. Assuming $f \geq 0$, the *shadow* of f is the subset of $\mathbb{R}^{n+1}$ described by

$$\text{Shad}(f) = \{(x, y) \mid x \in I, \quad 0 \leq y \leq f(x)\}.$$

Here is a sketch:



The goal of this section is to investigate the validity of the equation

$$\int_I f \;=\; \mathrm{vol}_{n+1}(\mathrm{Shad}(f)).$$

PROBLEM 9–42. Prove that $\underline{\mathrm{vol}}(\mathrm{Graph}(f)) = 0$.
(HINT: show that any special rectangle I which is contained in $\mathrm{Graph}(f)$ must have edge length zero in the x_{n+1} direction.)

The following problem shows that this equation for the inner volume of $\mathrm{Graph}(f)$ is as good as we can hope for without further assumptions.

PROBLEM 9–43. Here is an example I learned from Michael Boshernitzan. Define a function $[0,1] \xrightarrow{f} [0,1]$ by the following:

if x is a finite decimal, $x = 0.a_1a_2\dots a_N$, where $a_N \neq 0$, then $f(x) = 0.a_{N-1}\dots a_2a_1$; otherwise $f(x) = 0$.

E.g., $f(.285) = .82$, $f(.0025) = .2$, $f(.7) = 0$. Prove that $\mathrm{Graph}(f)$ is *dense* in the square $[0,1] \times [0,1]$. Then prove that

$$\overline{\mathrm{vol}}_2(\mathrm{Graph}(f)) \;=\; 1.$$

The next problem presents the verification that our desired equation is true (and easy) for step functions. The rest of the section will be devoted to extending this result to the general case.

PROBLEM 9–44. Let $I \xrightarrow{\varphi} \mathbb{R}$ be a nonnegative step function. Prove that $\text{Shad}(\varphi)$ is a special polygon, and that

$$\int_I \varphi = \text{vol}_{n+1}(\text{Shad}(\varphi)).$$

The theorem we are going to present is naturally divisible into two stages, so I choose to present them separately.

LEMMA 1. $\underline{\int} f = \underline{\text{vol}}_{n+1}(\text{Shad}(f))$.

PROOF. First we use the lemma of Section F3 applied to the indicator function of $\text{Shad}(f)$. Doing the x_{n+1}-integration first yields immediately from the first inequality of the lemma

$$\underline{\int} 1_{\text{Shad}(f)} \le \underline{\int}_I \left(\int_0^{f(x)} dx_{n+1} \right) dx;$$

that is,

$$\underline{\text{vol}}_{n+1}(\text{Shad}(f)) \le \underline{\int}_I f(x)dx;$$

Conversely, consider any step function σ which competes in the definition of the lower integral of f; we thus assume $0 \le \sigma \le f$. Then $y \le \sigma(x) \Rightarrow y \le f(x)$, so we conclude that

$$\text{Shad}(\sigma) \subset \text{Shad}(f).$$

We conclude from Problem 9–43 that

$$\begin{aligned} \int \sigma &= \text{vol}_{n+1}(\text{Shad}(\sigma)) \\ &\le \underline{\text{vol}}_{n+1}(\text{Shad}(f)). \end{aligned}$$

Since σ is arbitrary, the definition of the lower integral now implies that

$$\underline{\int} f \le \underline{\text{vol}}_{n+1}(\text{Shad}(f)).$$

QED

LEMMA 2. $\overline{\int} f = \overline{vol}_{n+1}(\text{Shad}(f))$.

PROOF. Just as in the proof of Lemma 1, the inequality $\le$ follows immediately from the lemma of Section F3. Conversely, consider any step function $\tau \ge f$. Then

$$\text{Shad}(f) \subset \text{Shad}(\tau),$$

and the rest of the proof is just like that of Lemma 1, yielding

$$\overline{\text{vol}}_{n+1}(\text{Shad}(f)) \ \leq \ \overline{\int} f.$$

QED

We therefore immediately conclude the truth of

THEOREM. *Assume* $I \xrightarrow{f} [0, \infty)$ *is a bounded function. Then*

$$\underline{\int} f \ = \ \underline{\text{vol}}_{n+1}(\text{Shad}(f)),$$
$$\overline{\int} f \ = \ \overline{\text{vol}}_{n+1}(\text{Shad}(f)).$$

Therefore, f *is integrable* $\Longleftrightarrow$ Shad(f) *is a contented set; moreover, in this case*

$$\int_I f \ = \ \text{vol}_{n+1}(\text{Shad}(f)).$$

This theorem is really wonderful in that it connects integrability of a function on $\mathbb{R}^n$ with contentedness of a set in $\mathbb{R}^{n+1}$. While we might have expected that the criterion would be expressed in terms of the *graph* of the function, the following three problems belie such an expectation.

PROBLEM 9–45. Assume $I \xrightarrow{f} \mathbb{R}$ is bounded. Prove that

$$\overline{\text{vol}}_{n+1}(\text{Graph}(f)) \ \leq \ \overline{\int} f - \underline{\int} f.$$

(HINT: why may you assume $f \geq 0$? For any $0 \leq \sigma < f \leq \tau$, show that

$$\text{Graph}(f) \subset \text{Shad}(\tau) - \text{Shad}(\sigma).)$$

PROBLEM 9–46. Let f be the standard function defined in Problem 9–6. Prove that the inequality of Problem 9–44 is strict for this f.

PROBLEM 9–47. Prove that if f is integrable, then Graph(f) is contented and has volume 0. Prove that the converse statement is false.

PROBLEM 9–48. In fact, prove that if f is the indicator function of *any* bounded set, then Graph(f) has volume 0.

PROBLEM 9–49. Let $I \xrightarrow{f} [0,\infty)$ be bounded. Show that the theorem of this section is valid if we instead define Shad(f) to be the set

$$\{(x,y) \mid 0 \le y < f(x)\}.$$

In particular, for any bounded function $I \xrightarrow{f} [0,\infty)$ the two sets

$$\{(x,y) \mid 0 \le y < f(x)\}, \quad \{(x,y) \mid 0 \le y \le f(x)\}$$

have the same outer volume and have the same inner volume.