# Chapter 14 Gauss' theorem

We now present the third great theorem of integral vector calculus. It is interesting that Green's theorem is again the basic starting point. In Chapter 13 we saw how Green's theorem directly translates to the case of surfaces in $\mathbb{R}^3$ and produces Stokes' theorem. Now we are going to see how a reinterpretation of Green's theorem leads to Gauss' theorem for $\mathbb{R}^2$, and then we shall learn from that how to use the *proof* of Green's theorem to extend it to $\mathbb{R}^n$; the result is called Gauss' theorem for $\mathbb{R}^n$.

A. **Green's theorem reinterpreted**

We begin with the situation obtained in Section 12C for a region $R$ in $\mathbb{R}^2$. With the positive orientation for bd$R$, we have

$$\iint\limits_R \frac{\partial f}{\partial x} dxdy = \int\limits_{\mathrm{bd}R} f dy,$$
$$\iint\limits_R \frac{\partial g}{\partial y} dxdy = -\int\limits_{\mathrm{bd}R} g dx.$$

We now consider a vector field $F = (F_1, F_2)$ on $R$, and we obtain from Green's theorem

$$\iint\limits_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dxdy = \int\limits_{\mathrm{bd}R} F_1 dy - F_2 dx.$$

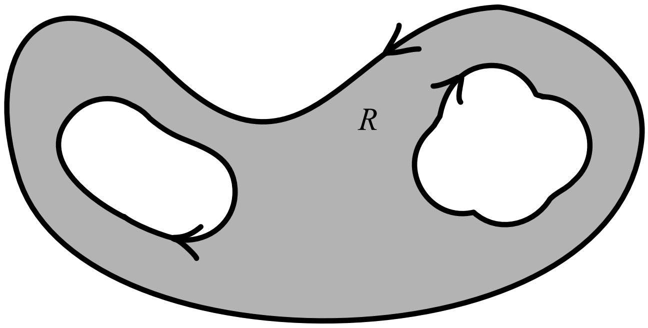
(Notice if we were thinking in terms of Stokes' theorem, we would put the minus sign on the left side instead of the right.)

We now work on the line integral, first writing it symbolically in the form

$$\int\limits_{\mathrm{bd}R} F \bullet (dy, -dx).$$

We now express what this actually means. Of course, bd$R$ may come in several disjoint pieces, so we focus on just one such piece, say the closed curve $\gamma = \gamma(t) = (\gamma_1(t), \gamma_2(t))$, $a \le t \le b$. Then the part of the line integral
corresponding to this piece is actually

$$\int_a^b F(\gamma(t)) \bullet (\gamma_2'(t), -\gamma_1'(t))dt.$$



This in turn equals

$$\int_a^b F(\gamma(t)) \bullet \frac{(\gamma_2'(t), -\gamma_1'(t))}{\|\gamma'(t)\|} \, \|\gamma'(t)\| dt.$$

The unit vector in the dot product has a beautiful geometric interpretation. Namely, it is orthogonal to the unit tangent vector

$$\frac{(\gamma_1'(t), \gamma_2'(t))}{\|\gamma'(t)\|},$$

and it is oriented 90° *clockwise* from the tangent vector. (You can use the matrix $J$ of Section 8D to see this, as

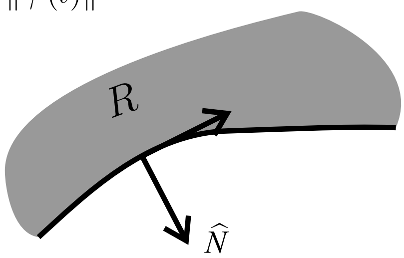
$$J \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} = \begin{pmatrix} -\gamma_2' \\ \gamma_1' \end{pmatrix}.$$

You can also think in complex arithmetic: $-i(\gamma_1' + i\gamma_2') = \gamma_2' - i\gamma_1'$, and multiplication by $-i$ rotates 90° clockwise.)

Because the tangent direction $\gamma'(t)$ has the region $R$ placed 90° *counterclockwise* from it, we conclude that the normal vector

$$\frac{(\gamma_2'(t), -\gamma_1'(t))}{\|\gamma'(t)\|}$$

points *away from* $R$:




Let us name this unit normal vector $\widehat{N}$. Thus at each point $(x, y) \in \text{bd}R$, the vector $\widehat{N}(x, y)$ at that point is determined by these requirements:

- $\widehat{N}$ has unit norm,
- $\widehat{N}$ is orthogonal to bd$R$,
- $\widehat{N}$ points outward from $R$.

Green's theorem has now been rewritten in the form

$$\iint\limits_R \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dxdy = \int\limits_{\text{bd}R} F \bullet \widehat{N} d\text{vol}_1.$$

This result is precisely what is called Gauss' theorem in $\mathbb{R}^2$. The integrand in the integral over $R$ is a special function associated with a vector field in $\mathbb{R}^2$, and goes by the name the *divergence* of $F$:

$$\text{div}F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

Again we can use the symbolic "del" vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

to write

$$\text{div}F = \nabla \bullet F.$$

Thus Gauss' theorem asserts that

$$\iint\limits_R \nabla \bullet F d\text{vol}_2 = \int_R F \bullet \widehat{N} d\text{vol}_1.$$

*Notice that this final result has the interesting feature that the orientation of the coordinate system has completely disappeared.*

B. **Gauss' theorem for** $\mathbb{R}^n$

It is very easy now to imagine what the correct extension to $\mathbb{R}^n$ should be. This is very different from our encounter with Stokes' theorem, for which the final result seems to be very much tied to $\mathbb{R}^3$.

Suppose $D \subset \mathbb{R}^n$ is an open set with a "reasonable" boundary bd$D$. We don't want to explain this further, except to say that at each point of bd$D$ there is a normal vector $\widehat{N}$ such that

- $\widehat{N}$ has unit norm,
- $\widehat{N}$ is orthogonal to bd$D$,
- $\widehat{N}$ points outward from $D$.

We are willing to allow corners and edges to some extent, but nothing too "wild" is to be permitted.

Suppose $F$ is a vector field of class $C^1$ defined at least on cl$D$, the closure of $D$. We define the *divergence* of $F$ by the formula

$$\text{div}F = \nabla \bullet F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

This is again the formal dot product

$$\nabla \bullet F \;=\; \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \bullet (F_1, \dots, F_n).$$

**GAUSS' THEOREM.** *Under the above hypotheses,*

$$\boxed{\int_D \nabla \bullet F d\mathrm{vol}_n \;=\; \int\limits_{bdD} F \bullet \widehat{N} d\mathrm{vol}_{n-1}.}$$

We shall spend the remainder of this section discussing examples of the use of this theorem, and shall give the proof in the next section.

First, apply the theorem to the very particular vector field $F(x) = x$, so that $\nabla \bullet F = n$. Then we obtain immediately

$$n\mathrm{vol}_n(D) \;=\; \int\limits_{\mathrm{bd}D} x \bullet \widehat{N} d\mathrm{vol}_{n-1}.$$

In particular, if $D$ is the ball $B(0, r)$ of radius $r$, we know $\mathrm{vol}_n(D) = \alpha_n r^n$. And $\mathrm{bd}D =$ the sphere $S(0, r)$, for which $\widehat{N} = x/r$. Thus $x \bullet \widehat{N} = \|x\|^2/r = r$, so we obtain immediately from Gauss' theorem the equation

$$\mathrm{vol}_{n-1}(S(0, a)) \;=\; n\alpha_n r^{n-1}.$$

This formula was found with more effort in Section 11C.

For another example we examine an ellipsoid. Let $A$ be a symmetric positive definite $n \times n$ matrix, and $D$ the "solid" ellipsoid

$$D \;=\; \{x \in \mathbb{R}^n \mid Ax \bullet x < 1\}.$$

We know from Problem 10–42 that

$$\mathrm{vol}_n(D) \;=\; \frac{\alpha_n}{\sqrt{\det A}}.$$

We also have the gradient

$$\nabla Ax \bullet x \;=\; 2Ax,$$

so that in Gauss' theorem

$$x \bullet \widehat{N} \;=\; \frac{Ax \bullet x}{\|Ax\|} \;=\; \frac{1}{\|Ax\|}.$$

Thus we obtain

$$\int\limits_{\mathrm{bd}D} \frac{d\mathrm{vol}_{n-1}}{\|Ax\|} = \frac{\alpha_n}{\sqrt{\det A}}.$$

In particular, if $A$ is the diagonal matrix with entries $a_1^{-2}, \dots, a_n^{-2}$, we obtain

$$\int\limits_{\frac{x_1^2}{a_1^2}+\cdots+\frac{x_n^2}{a_n^2}=1} \frac{d\mathrm{vol}_{n-1}}{\sqrt{\frac{x_1^2}{a_1^4}+\cdots+\frac{x_n^2}{a_n^4}}} = \alpha_n a_1 \dots a_n,$$

a result obtained in Problem 11–28 with much more effort required.

**PROBLEM 14–1.** Prove that if $f$ is a real-valued function and $F$ is a vector field, then the *product rule* is valid:

$$\nabla \bullet (fF) = f\nabla \bullet F + \nabla f \bullet F.$$

**PROBLEM 14–2.** Prove these special relations for $\mathbb{R}^3$:

$$\begin{aligned} \text{divergence of a curl} &= 0, \quad \text{i.e.} \\ \nabla \bullet (\nabla \times F) &= 0, \end{aligned}$$

and

$$\nabla \bullet (F \times G) = G \bullet \nabla \times F - F \bullet \nabla \times G.$$

Perhaps the most important use of the Gauss theorem is that it affords us a geometric interpretation of divergence. This interpretation is in the same spirit as our previous discussions of *gradient*, in Section 2F, "Geometric significance of the gradient," and of *curl*, in Section 13G, "What *is* curl?"

We approach this by first understanding the geometry of the integral over bd$D$ in the right side of Gauss' theorem. The terminology that is used derives from applications to fluid flow or electric fields. If $F$ is a vector field in $\mathbb{R}^n$ and $M$ is a hypermanifold for which a preferred "side" has been chosen by naming a continuous unit normal vector $\widehat{N}$, then $F \bullet \widehat{N}$ is the component of $F$ orthogonal to $M$, with a definite sign affixed. Then the integral

$$\int_M F \bullet \widehat{N} d\mathrm{vol}_{n-1}$$

in a sense measures the net amount of the vector field that "crosses" $M$. This is usually called the net *flux* of $F$ across $M$.

Then the right side of Gauss' theorem, the integral

$$\int_{\mathrm{bd}D} F \bullet \widehat{N} d\mathrm{vol}_{n-1},$$

represents the net flux of the vector field in the direction *outward* from $D$ across its boundary. Thus Gauss' theorem becomes in words,

$$\int_D \nabla \bullet F d\mathrm{vol}_n = \text{flux of } F \text{ out from } D.$$

Now let $x_0 \in \mathbb{R}^n$ be a fixed point, and choose $D$ to be the ball $B(x_0, \epsilon)$ with center $x_0$ and radius $\epsilon$. If $\epsilon$ is small, we have a good approximation

$$\int_{B(x_0,\epsilon)} \nabla \bullet F d\mathrm{vol}_n \approx \nabla \bullet F(x_0) \times \mathrm{vol}_n(B(x_0, \epsilon)).$$

Thus we obtain the formula

$$\nabla \bullet F(x_0) = \lim_{\epsilon \to 0} \frac{1}{\mathrm{vol}_n(B(x_0, \epsilon))} \int_{S(x_0,\epsilon)} F \bullet \widehat{N} d\mathrm{vol}_{n-1},$$

which we express in words as

$$\nabla \bullet F(x_0) = \text{the flux of } F \text{ away from } x_0 \text{, per unit volume.}$$

Hence the word "divergence."

Once again, we have an amazing situation that although $\nabla \bullet F$ was originally defined in a way that crucially required a coordinate system, the fact is that the quotient

$$\frac{1}{\mathrm{vol}_n(B(x_0, \epsilon))} \int_{S(x_0,\epsilon)} F \bullet \widehat{N} d\mathrm{vol}_{n-1}$$

manifestly is independent of any coordinate system; therefore, its limit as $\epsilon \to 0$, namely the divergence of $F$ at $x_0$, is also completely determined geometrically.

PROTOEXAMPLE. This example in case $n = 3$ is the force field of a unit electric charge placed at the origin, or the gravitational field of a unit mass at the origin. In the case $n = 3$ it's called the "inverse square law." For general $n$, we have to within a constant factor

$$F(x) = \frac{x}{\|x\|^n}.$$

Then we calculate from the product rule of Problem 14–1,

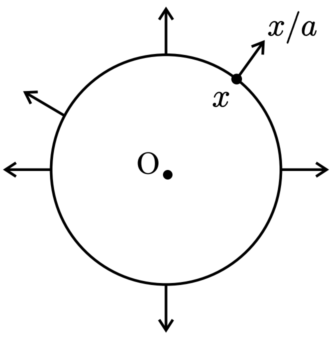
$$\begin{aligned}\nabla \bullet F &= \frac{\nabla \bullet x}{\|x\|^n} + \nabla(\|x\|^{-n}) \bullet x \\ &= \frac{n}{\|x\|^n} + (-n\|x\|^{-n-2}x) \bullet x \\ &= \frac{n}{\|x\|^n} - n\|x\|^{-n-2}\|x\|^2 \\ &= 0.\end{aligned}$$

Of course we notice the crucial role here of the power $n$. It comes from the fact that on $\mathbb{R}^n$ we have $\nabla \bullet x = n$. Thus on $\mathbb{R}^n$ our vector field is

$$F(x) = \frac{x/\|x\|}{\|x\|^{n-1}},$$

so it's an "inverse $(n-1)$ power law."

Now we compute an easy case, the outward flux of $F$ across the sphere $S(0,a)$. As the outward unit normal is $\widehat{N} = x/a$, we find




$$\begin{aligned}\int_{S(0,a)} F \bullet \widehat{N} d\mathrm{vol}_{n-1} &= \int_{S(0,a)} \frac{x}{a^n} \bullet \frac{x}{a} d\mathrm{vol}_{n-1} \\ &= \int_{S(0,a)} \frac{a^2}{a^{n+1}} d\mathrm{vol}_{n-1} \\ &= \mathrm{vol}_{n-1}(S(0,1)) \\ &= n\alpha_n.\end{aligned}$$

NOTATION. In the present investigations the number $n\alpha_n$ is going to be appearing quite often. Let us denote it as

$$\begin{aligned}\omega_n &= n\alpha_n \\ &= \mathrm{vol}_{n-1}(S(0,1)).\end{aligned}$$

That is, $\omega_n$ is the measure of the unit sphere in $\mathbb{R}^n$. Thus

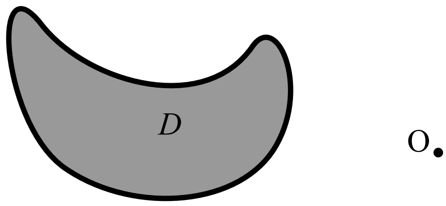
$$\begin{aligned}\omega_2 &= 2\pi, \\ \omega_3 &= 4\pi, \\ \omega_4 &= 2\pi^2,\end{aligned}$$

and in general

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \qquad \text{(see p. 11–15).}$$

Now suppose $D$ is a region in $\mathbb{R}^n$ such that the origin is exterior to it; that is, $0 \notin \mathrm{cl}D$. Since $F$ is well defined in $\mathrm{cl}D$ and has zero divergence, Gauss' theorem implies immediately that

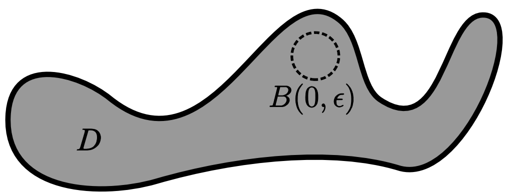
$$\int_{\mathrm{bd}D} F \bullet \widehat{N} d\mathrm{vol}_{n-1} = 0.$$




On the other hand, suppose the origin is in the interior of $D$: $0 \in D$. Then we can prove that

$$\int_{\mathrm{bd}D} F \bullet \widehat{N} d\mathrm{vol}_{n-1} = \omega_n.$$

This is seen very easily by using what is often called a "safety ball." That is a ball $B(0,\epsilon)$ whose radius is so small that $\mathrm{cl}B(0,\epsilon) \subset D$.




Then $D - B(0,\epsilon)$ is "safe" in that the origin is not in it, so we have from the preceding case

$$\int_{\mathrm{bd}(D-B(0,\epsilon))} F \bullet \widehat{N} d\mathrm{vol}_{n-1} = 0.$$

But $\mathrm{bd}(D - B(0,\epsilon))$ is composed of the two disjoint portions $\mathrm{bd}D$ and the sphere $S(0,\epsilon)$. Thus

$$\int_{\mathrm{bd}D} F \bullet \widehat{N} d\mathrm{vol}_{n-1} + \int_{S(0,\epsilon)} F \bullet \widehat{N} d\mathrm{vol}_{n-1} = 0.$$

The latter integrand has $\widehat{N} = -x/\epsilon$, as $\widehat{N}$ is oriented outward relative to $D - B(0,\epsilon)$. Thus the latter integral is actually equal to $-\omega_n$. This proves the desired equation.

Another viewpoint of the above analysis is to move the center of the field from the origin to an arbitrary point $x_0$ by translation. Then we obtain

$$\int_{\mathrm{bd}D} \frac{x - x_0}{\|x - x_0\|^n} \bullet \widehat{N} d\mathrm{vol}_{n-1} = \begin{cases} \omega_n & \text{if } x_0 \in D, \\ 0 & \text{if } x_0 \notin \mathrm{cl}D. \end{cases}$$

The case $x_0 \in \mathrm{bd}D$ does not fall in either category.

**PROBLEM 14–3.** One might guess that for $x_0 \in \mathrm{bd}D$

$$\int_{\mathrm{bd}D} \frac{x - x_0}{\|x - x_0\|^n} \bullet \widehat{N} d\mathrm{vol}_{n-1} = \frac{1}{2}\omega_n.$$

Show that this is indeed the case if $n = 2$ and 3 and $D$ is a "ball" (disk if $n = 2$).

**PROBLEM 14–4.** Show that the result of the preceding problem is valid for all $n \geq 2$.

NAMES. In Russian texts Gauss' theorem is called *Ostrogradski's* theorem. It's also frequently called the *divergence* theorem.

C. **The proof**

In one sense the proof we give is a generalization of the proof of Green's theorem as given in Chapter 12. The basic idea of integrating first in the direction of the partial derivative in the integrand is still the key ingredient of the proof. That is, we want to prove that for each fixed index $i$

$$\int_D \frac{\partial F_i}{\partial x_i} d\mathrm{vol}_n = \int_{\mathrm{bd}D} F_i \widehat{N} \bullet \hat{e}_i d\mathrm{vol}_{n-1},$$

and so we approach this by performing the $x_i$-integration on the left side before the other integrations. We then sum for $i = 1, \dots, n$, to obtain the theorem. In our analysis we may as well replace the component $F_i$ with just a real-valued function $f$ of class $C^1$. Thus we need to prove

$$\int_D \frac{\partial f}{\partial x_i} d\mathrm{vol}_n = \int_{\mathrm{bd}D} f N_i d\mathrm{vol}_{n-1}, \qquad (*)$$

where $N_i = \widehat{N} \bullet e_i$ is the $i^{\text{th}}$ component of the outer unit normal vector $\widehat{N}$.

In another sense the proof we present here is much more efficient than the one we gave in Chapter 12, as we employ a beautiful technical device to get around the troublesome details about patching results together (see Section 12C). This technique is called a *partition of unity.*

Here is a brief description of it. Suppose $\epsilon > 0$ is given. Then there is a set of $C^1$ functions $\psi_1, \dots, \psi_N$ defined on $\mathbb{R}^n$ such that

$$\begin{aligned} \sum_{k=1}^{N} \psi_k &= 1 \quad \text{on } \operatorname{cl} D, \\ \text{for each } k,\ \psi_k &= 0 \text{ outside some ball of radius } \epsilon. \end{aligned}$$

This is easier to prove than you might imagine. One first chooses nonnegative $\psi_k$'s with the second property, such that their sum is positive on cl$D$, and then each is normalized by dividing by their sum. Incidentally, instead of $C^1$ we could use $C^2$ or $C^3$ etc., or even $C^\infty$.

Using such a partition of unity, it suffices to prove $(*)$ for each product $\psi_k f$ and then add the results to achieve the theorem. Another way of expressing this is to say that we may assume *ab initio* that $f$ itself is zero except on some small ball.

There are now two separate cases to handle. The first is the case in which the ball where $f$ is nonzero is contained in (the interior of) the open set $D$. Then the right side of $(*)$ is zero since its integrand $fN_i$ equals zero on bd$D$. But also the left side is zero, as we can write

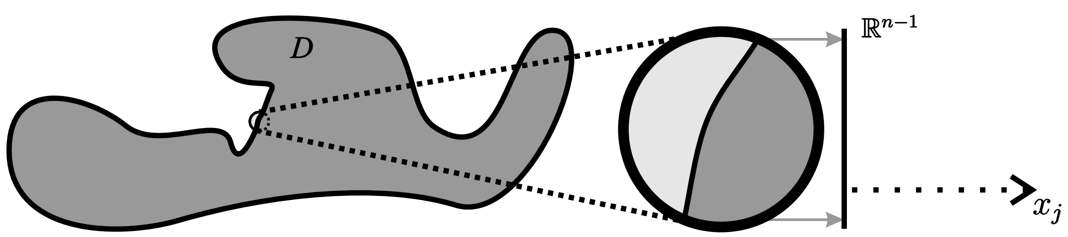
$$\begin{aligned} \int_D \frac{\partial f}{\partial x_i} d\mathrm{vol}_n &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i} d\mathrm{vol}_n \\ &\overset{\text{Fubini}}{=} \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_i} dx_i \right) d\mathrm{vol}_{n-1} \end{aligned}$$

and the inner integral

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_i} dx_i = 0$$

by the fundamental theorem of calculus.

That was easy. The second case is not much harder but is very much more interesting. In this case $f$ is nonzero only in a small ball, but this ball intersects bd$D$. Here we explain the philosophy behind our partition of unity. We choose the balls so small that when one of them intersects bd$D$ the open set $D$ near that ball has a particular sort of description. Namely, it can be represented as "one side" of a graph of a $C^1$ function. For instance, consider the following typical situation:



The portion of $D$ in the illustrated ball is described by an inequality of the form

$$x_j \;>\; \varphi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

(If the ball were on the "right side" of the picture, the defining inequality would appear as

$$x_j \;<\; \varphi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

We content ourselves with the first case in the proof.)

We pause to streamline some of the notation. First, since $f$ is zero outside the small ball, we may as well assume $f$ is defined for all $x_j$ satisfying the given inequality.

Second, we may as well arrange coordinates so that $j = n$. This merely has the effect of greatly shortening the notation.

Third, we denote by $x' = (x_1, \ldots, x_{n-1})$ so that $f$ is defined for all points $x$ satisfying

$$x_n \;>\; \varphi(x'),\; x' \in R,$$

where $R \subset \mathbb{R}^{n-1}$ is some small ball.

We now have formulas for $\widehat{N}$ and $d\text{vol}_{n-1}$. Namely, since $\widehat{N}$ is the outward unit normal,

$$\widehat{N} \;=\; \frac{(\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_{n-1}, -1)}{\sqrt{\|\nabla\varphi\|^2 + 1}},$$

and

$$\begin{aligned} d\text{vol}_{n-1} \;&=\; \sqrt{\|\nabla\varphi\|^2 + 1}\; dx_1 \;\ldots\; dx_{n-1} \\ &=\; \sqrt{\|\nabla\varphi\|^2 + 1}\; dx'. \end{aligned}$$

Thus

$$N_i d\text{vol}_{n-1} \;=\; \begin{cases} \partial\varphi/\partial x_i dx' & \text{if } \; i \neq n, \\ -dx' & \text{if } \; i = n. \end{cases}$$

These formulas require us to finish with two different cases.

$\boxed{i = n}$ In this easier case we have

$$\begin{aligned} \int_D \frac{\partial f}{\partial x_n} d\text{vol}_n \;&\overset{\text{Fubini}}{=}\; \int_R \int_{\varphi(x')}^{\infty} \frac{\partial f}{\partial x_n} dx_n dx' \\ &\overset{\text{FTC}}{=}\; \int_R f \,\Big|_{x_n=\varphi(x')}^{x_n=\infty} dx' \\ &=\; \int_R -f(x', \varphi(x'))dx' \\ &=\; \int_{\text{bd}D} f N_n d\text{vol}_{n-1}, \end{aligned}$$

and $(*)$ is proved.

$\boxed{i \neq n}$ Here we have to face the problem that doing the $x_i$ integration first may not be so feasible. We need to do the $x_n$ integration first, so for each fixed $x' \in R$ we shall change variables by setting

$$x_n = \varphi(x') + t.$$

Then $0 < t < \infty$ and we have

$$\begin{aligned}
\int_D \frac{\partial f}{\partial x_i} d\mathrm{vol}_n &\overset{\text{Fubini}}{=} \int_R \int_{\varphi(x')}^{\infty} \frac{\partial f}{\partial x_i}(x', x_n) dx_n dx' \\
&= \int_R \int_0^{\infty} \frac{\partial f}{\partial x_i}(x', \varphi(x') + t) dt dx' \\
&\overset{\text{Fubini}}{=} \int_{R\times(0,\infty)} \frac{\partial f}{\partial x_i}(x', \varphi(x') + t) dt dx' \\
&\overset{\text{chain rule}}{=} \int_{R\times(0,\infty)} \left[ \frac{\partial}{\partial x_i}\left(f(x', \varphi(x') + t)\right) - \frac{\partial \varphi}{\partial x_i}\frac{\partial f}{\partial x_n} \right] dt dx'.
\end{aligned}$$

We now have two integrations to perform. We handle the first by doing the $x_i$ integration first. The result is zero, as for each fixed $t$ we essentially have

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x_i}\left(f(x', \varphi(x') + t)\right) dx_i \overset{\text{FTC}}{=} 0.$$

In the second integral we return from $t$ to $x_n$. Thus we have

$$\begin{aligned}
\int_D \frac{\partial f}{\partial x_i} d\mathrm{vol}_n &= \int_{R\times(0,\infty)} -\frac{\partial \varphi}{\partial x_i}\frac{\partial f}{\partial x_n}(x', \varphi(x') + t) dt dx' \\
&= \int_R -\frac{\partial \varphi}{\partial x_i} \int_{\varphi(x')}^{\infty} \frac{\partial f}{\partial x_n}(x', x_n) dx_n dx' \\
&\overset{\text{FTC}}{=} \int_R -\frac{\partial \varphi}{\partial x_i} f(x', x_n) \Big|_{x_n = \varphi(x')}^{x_n = \infty} dx' \\
&= \int_R \frac{\partial \varphi}{\partial x_i} f(x', \varphi(x')) dx' \\
&= \int_{\mathrm{bd}D} f N_i d\mathrm{vol}_{n-1}.
\end{aligned}$$

This finishes the proof of Gauss' theorem.

D. **Gravity**

The applications of Gauss' theorem are quite diverse and important. Especially in the rudiments of electricity and magnetism and in gravitation we find the theorem playing an important role. We content ourselves here with some observations on gravitation.

Isaac Newton's theory includes as an axiom that two point masses in $\mathbb{R}^3$ attract one another with an inverse square force. We of course are determined to generalize this to $\mathbb{R}^n$, where it presumably makes no physical sense. So suppose we have a mass $m_1$ located at $x \in \mathbb{R}^n$ and another one $m_2$ located at $y \in \mathbb{R}^n$. Then the force at $y$ due to the mass at $x$ is equal to the vector

$$Gm_1 m_2 \, \frac{x-y}{\|x-y\|^n},$$

where the number $G > 0$ is called the "gravitational constant." We shall assume from now on that $G = 1$.

One of the first things that can be proved has to do with a mass distributed uniformly over a sphere. We can set this situation up in the following way. Let us suppose the sphere is centered at the origin, so that it equals $S(0,a)$, where $a$ is its radius. Let us suppose the mass spread over this sphere is $M$. We say the mass *density* is the constant

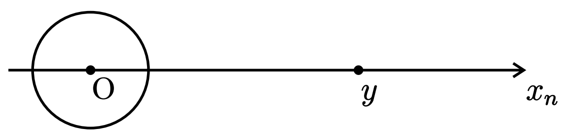
$$\frac{M}{\mathrm{vol}_{n-1}(S(0,a))} = \frac{M}{\omega_n a^{n-1}}.$$

Now suppose that we have a unit point mass located at $y$. Then according to Newton the force at $y$ due to the mass spread over the sphere is given by the integral

$$\int_{S(0,a)} \frac{x-y}{\|x-y\|^n} \, \frac{M}{\omega_n a^{n-1}} \, d\mathrm{vol}_{n-1}(x).$$

What we are going to do now is calculate this integral.

In doing the calculation we may choose coordinates to suit ourselves, so let us place the positive $n^{\text{th}}$ coordinate direction pointing toward $y$. In other words, we assume that $y = r\hat{e}_n$, where of course $r = \|y\|$.



By symmetry we notice that the above integral, which is a vector, must have zero components in the directions orthogonal to $\hat{e}_n$. Thus we may as well just compute the $n^{\text{th}}$ coordinate of the integral, that is

$$\frac{M}{\omega_n a^{n-1}} \int_{S(0,a)} \frac{x_n - r}{\|x - r\hat{e}_n\|^n} \, d\text{vol}_{n-1}(x).$$

THE PHYSICAL CASE $n = 3$. It is quite impressive that the easiest dimension for performing the above integral is $n = 3$. We now carry this out. First, we use the familiar result

$$\|x - y\|^2 \;=\; \|x\|^2 - 2x \bullet y + \|y\|^2$$

to rewrite the denominator, obtaining

$$\frac{M}{4\pi a^2} \int_{S(0,a)} \frac{x_3 - r}{(a^2 - 2rx_3 + r^2)^{3/2}} \, d\text{vol}_2(x).$$

Next use the usual spherical coordinates with $x_3 = a\cos\varphi$, to obtain

$$\begin{aligned} &\frac{M}{4\pi a^2} \cdot 2\pi \int_0^\pi \frac{a\cos\varphi - r}{(a^2 - 2ar\cos\varphi + r^2)^{3/2}} \, a^2 \sin\varphi d\varphi \\ &= \frac{M}{2} \int_0^\pi \frac{a\cos\varphi - r}{(a^2 - 2ar\cos\varphi + r^2)^{3/2}} \, \sin\varphi d\varphi. \end{aligned}$$

**PROBLEM 14–5.** Physics texts know what to do here. Make the substitution

$$u \;=\; a^2 - 2ar\cos\varphi + r^2$$

to change the dummy $\varphi$ into $u$, and thus show that the above integral equals

$$\begin{cases} -M/r^2 & \text{if } \; r > a, \\ -M/2a^2 & \text{if } \; r = a, \\ 0 & \text{if } \; 0 \le r < a. \end{cases}$$

The physical interpretation of this result is well known and extremely important. Namely, at points outside the closed ball $\overline{B}(0,a)$ the gravitational attraction of the uniform mass spread over the sphere is the same as if all the mass were concentrated at the center of the sphere. At points in the interior $B(0,a)$ of the ball, the gravitational attraction is zero. And at points on the sphere $S(0, a)$ itself, the gravitational attraction is the same as if all the mass were concentrated at the center of the sphere, divided by 2.

**PROBLEM 14–6.** (not recommended) Repeat all of the above for the case $n = 2$. The integration is much more difficult; a table of integrals might be helpful.

At this point you should be wondering what any of this has to do with Gauss' theorem. We now explain this. The direct calculation of the integral on the preceding page is quite a formidable task. The case $n = 3$ is not so bad, but the case $n = 2$ is extremely difficult, and one can imagine that the situation for general $n$ could be even worse. Now we are going to show how a combination of some beautiful Euclidean geometry and Gauss' theorem enable us to calculate the integral very easily.

INVERSION. The geometry we have just referred to is called *inversion in a sphere*. (See pp. 6–34 and 7–23.) Suppose that $S(0, a)$ is a sphere in $\mathbb{R}^n$ and $y \in \mathbb{R}^n$. As you will see, we must assume $y \neq 0$. We then say that the *inversion* of $y$ is the point $y^\star$ which is uniquely defined by requiring that it lie on the ray from 0 through $y$ and that $\|y\|\|y^\star\| = a^2$. In other words,

$$y^\star = \frac{a^2}{\|y\|^2} y.$$

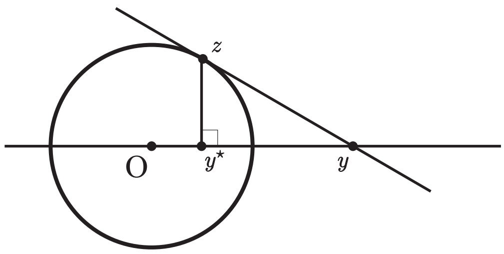
Notice that

$$y^{\star\star} = y$$

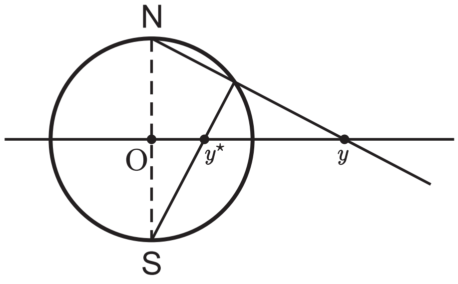
and that

$$y^\star = y \Longleftrightarrow y \in S(0, a).$$

**PROBLEM 14–7.** Verify the following picture: we assume $\|y\| > a$ and that a line has been drawn from $y$ tangent to the sphere at the point $z$. The picture is then in the unique two-dimensional plane determined by 0, $y$, and $z$.



**PROBLEM 14–8.** As in the preceding problem, verify this similar two-dimensional depiction of the relation between $y$ and $y^\star$.




Now we notice a very interesting phenomenon. If $x \in S(0,a)$, then

$$\begin{aligned}\|x-y^\star\|^2 &= \|x\|^2 - 2x \bullet y^\star + \|y^\star\|^2 \\ &= a^2 - 2x \bullet \frac{a^2 y}{\|y\|^2} + \frac{a^4}{\|y\|^2} \\ &= \frac{a^2}{\|y\|^2}\left(\|y\|^2 - 2x \bullet y + a^2\right) \\ &= \frac{a^2}{\|y\|^2}\left(\|y\|^2 - 2x \bullet y + \|x\|^2\right) \\ &= \frac{a^2}{\|y\|^2}\|y-x\|^2.\end{aligned}$$

That is,

$$\|x-y^\star\| \;=\; \frac{a}{\|y\|}\,\|x-y\| \qquad \text{for all } \; x \in S(0,a).$$

This equation is quite interesting, in that it shows that the ratio $\|x-y^\star\|/\|x-y\|$ is *constant* in its dependence on $x \in S(0,a)$.

We now return to the calculation of the integral at the bottom of p. 14–12. We of course may assume $y \neq 0$, since when $y = 0$ the integral equals zero thanks to the oddness of the integrand. By symmetry it is clear that the vector represented by this integral has zero components in directions orthogonal to $y$; in other words,

$$\frac{M}{\omega_n a^{n-1}} \int_{S(0,a)} \frac{x-y}{\|x-y\|^n}\, d\mathrm{vol}_{n-1}(x) \;=\; -cy,$$

where $c$ is a scalar. In other words, the gravitational attraction at $y$ points toward the origin. As we just need to calculate $c$, let us form the dot product of each side of the equation with $y$, denoting $r = \|y\|$. We obtain

$$\frac{M}{\omega_n a^{n-1}} \int_{S(0,a)} \frac{x \bullet y - r^2}{\|x-y\|^n} \, d\text{vol}_{n-1}(x) \; = \; -cr^2.$$

As we are wanting to apply the divergence theorem, we would like the integrand to be presented as the dot product of a vector field with $\widehat{N}$, which is equal to $x/a$. Therefore we write the numerator

$$\begin{aligned} x \bullet y - r^2 &= x \bullet y - \frac{r^2}{a^2} x \bullet x \\ &= \left( y - \frac{r^2}{a^2} x \right) \bullet x \\ &= \left( y - \frac{r^2}{a^2} x \right) \bullet a\widehat{N} \\ &= -\frac{r^2}{a} \left( x - \frac{a^2}{r^2} y \right) \bullet \widehat{N} \\ &= -\frac{r^2}{a} \, (x - y^\star) \bullet \widehat{N}. \end{aligned}$$

Aha! Notice how naturally the inversion $y^\star$ has appeared! We thus have found that

$$c \; = \; \frac{M}{\omega_n a^n} \int_{S(0,a)} \frac{x - y^\star}{\|x-y\|^n} \bullet \widehat{N} d\text{vol}_{n-1}(x).$$

Now we utilize the relationship between $\|x-y\|$ and $\|x-y^\star\|$ to find that

$$c \; = \; \frac{M}{\omega_n r^n} \int_{S(0,a)} \frac{x - y^\star}{\|x-y^\star\|^n} \bullet \widehat{N} d\text{vol}_{n-1}(x).$$

We find that we are in the situation of the protoexample of Section B. There are three cases.

$\boxed{\|y\| > a}$ In this case $\|y^\star\| < a$, so we obtain from p. 14–8 that

$$c \; = \; \frac{M}{r^n}.$$

$\boxed{\|y\| < a}$ In this case $\|y^\star\| > a$, so we obtain instead

$$c = 0.$$

$\boxed{\|y\| = a}$ In this case $\|y^\star\| = a$, so Problem 14–4 gives

$$c = \frac{M}{2a^n}.$$

In summary, the gravitational attraction of a uniform mass distribution over a sphere in $\mathbb{R}^n$ behaves in the exterior of the sphere as if all the mass were concentrated at the center, and is zero in the interior of the sphere. At points on the sphere itself, it is as if all the mass were concentrated at the center, divided by 2.

## E. **Other differentiation formulas**

**PROBLEM 14–9.** In analogy with the formula for curl$F$ given on p. 13–22, suppose that $\{\hat{\varphi}_1, \dots, \hat{\varphi}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, and represent points in coordinates relative to this basis as

$$x = t_1\hat{\varphi}_1 + \cdots + t_n\hat{\varphi}_n.$$

Then explain why

$$\text{div}F = \sum_{i=1}^{n} \frac{\partial}{\partial t_i}(F \bullet \hat{\varphi}_i).$$

**PROBLEM 14–10.** Explain why the formula just presented can also be written in terms of directional derivatives as

$$\text{div}F = \sum_{i=1}^{n} D(F \bullet \hat{\varphi}_i;\ \hat{\varphi}_i).$$

Then show that this formula remains true for a general orthogonal basis.

**DEFINITION.** Given a vector $A \in \mathbb{R}^n$, the first-order differential operator $A \bullet \nabla$ is defined by

$$(A \bullet \nabla)(f) = A \bullet \nabla f,$$

where $f$ is a real-valued differentiable function.

**REMARK.** Since both gradient and dot product have intrinsic meaning, so does this operator. In fact, it is clear that in terms of directional derivatives

$$(A \bullet \nabla)f(x) = Df(x; A).$$

**PROBLEM 14–11.** Show that for vector fields $F$ and $G$ on $\mathbb{R}^3$ we have

$$\nabla \times (F \times G) = (\nabla \bullet G)F - (\nabla \bullet F)G + (G \bullet \nabla)F - (F \bullet \nabla)G.$$

**PROBLEM 14–12.** Show that for a vector field $F$ on $\mathbb{R}^3$,

$$\nabla \times (\nabla \times F) = -\nabla^2 F + \nabla(\nabla \bullet F).$$

In this formula the *Laplacian* of $F$ is the vector field

$$\nabla^2 F = \sum_{i=1}^{3} \frac{\partial^2 F}{\partial x_i^2}.$$

## F. **The vector potential**

This section corresponds to Section 12F, where we considered the problem of finding a (real valued) potential for a vector field with zero curl. That is, given a vector field $F$ on $\mathbb{R}^3$ satisfying $\nabla \times F = 0$, we sought a potential function $f$ for which $F = \nabla f$. We based our thinking on the differentiation formula

$$\nabla \times \nabla f = 0$$

(see Section 13G).

The analogous formula we now investigate is the one which states

$$\nabla \bullet (\nabla \times G) = 0$$

for any $C^2$ vector field $G$ on $\mathbb{R}^3$ (see Problem 14–2). We are thus naturally led to the following question: if $F$ is a vector field on $\mathbb{R}^3$ for which $\nabla \bullet F = 0$, does there exist a vector field $G$ satisfying

$$F = \nabla \times G?$$

The answer is yes, provided that $F$ is a vector field defined on all of $\mathbb{R}^3$. The proof amounts to a rather logical procedure of integrating the above differential equation for $G$. Before doing this, note that there is a strong lack of uniqueness. For if $F = \nabla \times G_1$ and also $F = \nabla \times G_2$, then $0 = \nabla \times (G_1 - G_2)$. Thus from Section 12F we conclude that $G_1 - G_2 = \nabla f$ for some function $f$. Conversely, since $\nabla \times (\nabla f) = 0$, we can always add to $G$ *any* gradient field without changing the validity of $F = \nabla \times G$.

**DEFINITION.** A vector field $G$ satisfying the equation

$$F = \nabla \times G$$

is called a *vector potential* for the field $F$.

**THEOREM.** *Suppose $F$ is a vector field of class $C^1$ defined on all of $\mathbb{R}^3$, and suppose $\nabla \bullet F = 0$. Then there exists a vector potential for $F$; this vector potential is unique up to the addition of a gradient field.*

**PROOF.** The differential equations which $G$ must satisfy are

$$\begin{aligned}
\frac{\partial G_3}{\partial x_2} - \frac{\partial G_2}{\partial x_3} &= F_1, \\
\frac{\partial G_1}{\partial x_3} - \frac{\partial G_3}{\partial x_1} &= F_2, \\
\frac{\partial G_2}{\partial x_1} - \frac{\partial G_1}{\partial x_2} &= F_3.
\end{aligned}$$

This task seems daunting at first glance. However, we may as well assume $G_3 \equiv 0$, for we can find a function $f$ whose gradient equals

$$\nabla f = (?, ?, G_3)$$

by the simple device of writing

$$f(x) = \int_0^{x_3} G(x_1, x_2, t)dt.$$

Then $G - \nabla f$ has zero third component.

With $G_3 = 0$, the equations now become

$$\begin{aligned}
-\frac{\partial G_2}{\partial x_3} &= F_1, \\
\frac{\partial G_1}{\partial x_3} &= F_2, \\
\frac{\partial G_2}{\partial x_1} - \frac{\partial G_1}{\partial x_2} &= F_3.
\end{aligned}$$

Now choose $G_2$ to be any function satisfying the first of these equations. It remains to solve for $G_1$. Integrate the second equation to find $G_1$ to within an additive function $f(x_1, x_2)$:

$$G_1 = G_1^0 + f(x_1, x_2).$$

Then we need to find $f$ so that the third equation is satisfied. That is,

$$\frac{\partial f}{\partial x_2} = -F_3 - \frac{\partial G_1^0}{\partial x_2} + \frac{\partial G_2}{\partial x_1}.$$

There is no problem integrating this equation, *provided that its right side is independent of* $x_3$. We check this easily:

$$\begin{aligned}
\frac{\partial}{\partial x_3}\left(-F_3 - \frac{\partial G_1^0}{\partial x_2} + \frac{\partial G_2}{\partial x_1}\right) \\
&= -\frac{\partial F_3}{\partial x_3} - \frac{\partial^2 G_1^0}{\partial x_3 \partial x_2} + \frac{\partial^2 G_2}{\partial x_3 \partial x_1} \\
&= -\frac{\partial F_3}{\partial x_3} - \frac{\partial^2 G_1^0}{\partial x_2 \partial x_3} + \frac{\partial^2 G_2}{\partial x_1 \partial x_3} \\
&= -\frac{\partial F_3}{\partial x_3} - \frac{\partial F_2}{\partial x_2} - \frac{\partial F_1}{\partial x_1} \\
&= 0 \quad \text{by hypothesis.}
\end{aligned}$$

QED

**PROBLEM 14–13.** Show that the above procedure produces for a *constant* vector field $F$ the vector potential $G = (x_3F_2 - x_2F_3, -x_3F_1, 0)$.

**PROBLEM 14–14.** If $F$ is constant we earlier calculated that $\frac{1}{2}F \times \vec{x}$ is a suitable vector potential (see Problem 13–10). Find a function $f$ so that the solution of the preceding problem satisfies

$$G - \frac{1}{2}F \times \vec{x} = \nabla f.$$

**PROBLEM 14–15.** Using the notation of Problem 13–11, suppose $F(\vec{x}) = A\vec{x}$ is a linear vector field. Prove that $\nabla \bullet F = \text{trace}A$. Supposing $\text{trace}A = 0$, find a vector potential $G$ for $F$.

**PROBLEM 14–16.** Suppose $f$ and $g$ are real valued functions of class $C^2$ defined on an open set $D \subset \mathbb{R}^3$. Define the vector field

$$F = \nabla f \times \nabla g.$$

Prove that $\nabla \bullet F = 0$, and find a vector potential for $F$ which is defined on $D$. (HINT: compute $\nabla \times (f \nabla g)$.)

**PROBLEM 14–17.** Show that the vector field

$$F = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

has zero divergence on $\mathbb{R}^3-$ ($z$-axis). Show that it has a vector potential of the form $f(x, y)\hat{k}$, and calculate a suitable $f$ explicitly.

**PROBLEM 14–18*.** Show that the vector field

$$F = \left( \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0 \right)$$

has zero divergence on $\mathbb{R}^3-$ ($z$-axis), and find a vector potential for $F$ which is defined on all of $\mathbb{R}^3-$ ($z$-axis).

**PROBLEM 14–19.** (The analog of Section 12G.) Suppose that $F$ is a $C^1$ vector field defined on a *star shaped* open set $D \subset \mathbb{R}^3$. Suppose that $F$ has zero divergence. Supposing that $D$ is star shaped with respect to the origin, define the vector field

$$G(x) = \int_0^1 F(tx) \times tx dt.$$

Prove that

$$\nabla \times G = F.$$

**PROBLEM 14–20.** Use the preceding problem to give another solution of Problem 14–15, and show that the corresponding vector potential is

$$\frac{1}{3}(A\vec{x}) \times \vec{x}.$$

**PROBLEM 14–21.** Consider the gravitational vector field

$$F = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3-$ origin. It has zero divergence. Use the scheme of the general theorem of this section to find a vector potential. Show that you may reach an answer in the form

$$G = \frac{z}{(x^2 + y^2)\sqrt{x^2 + y^2 + z^2}} \quad (y, -x, 0).$$

**PROBLEM 14–22.** The solution which you obtained for the preceding problem may seem unsatisfactory, as it is not only badly behaved at the origin (necessarily), but also on the entire $z$ axis. Prove that in fact there is no vector potential for the gravitational field $F$ defined on all of $\mathbb{R}^3-$ origin.
(HINT: supposing such a potential $G$ would exist, compute

$$\iint_{\text{unit sphere}} F \bullet \widehat{N} d\text{area}$$

two different ways.)