Chapter 13 Stokes' theorem

In the present chapter we shall discuss $\mathbb{R}^3$ only. We shall use a right-handed coordinate system and the standard unit coordinate vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. We shall also name the coordinates x, y, z in the usual way.

The basic theorem relating the fundamental theorem of calculus to multidimensional integration will still be that of Green. In this chapter, as well as the next one, we shall see how to generalize this result in two directions. In this chapter we generalize it to surfaces in $\mathbb{R}^3$, whereas in the next chapter we generalize to regions contained in $\mathbb{R}^n$. But in all of these procedures it is still Green's theorem that is fundamental.

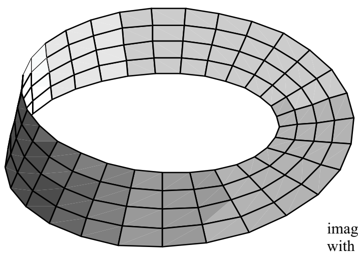
A. **Orientable surfaces**

We shall be dealing with a two-dimensional manifold $M \subset \mathbb{R}^3$. We'll just use the word *surface* to describe M. There are two features of M that we need to discuss first.

The first is the idea of a *normal vector* for M. We assume that M is of class C^1, so that at each point $p \in M$ there is a vector of unit norm which is orthogonal to M, in the sense that it is orthogonal to the tangent space T_pM. There are of course two choices of such a normal vector, and we now need to make a choice.

DEFINITION. The surface M is said to be *orientable* if there exists a unit normal vector $\widehat{N}(p)$ at each point $p \in M$ which is a continuous function of p.

The continuity of $\widehat{N}(p)$ is all-important. For instance, one can construct a Möbius strip and obtain a surface which is not orientable:



In case a surface is described implicitly by an equation

$$g(x, y, z) = 0$$

such that ∇g is never 0 at any point of the surface, and if g is of class C^1, then ∇g is continuous and we have two choices for $\widehat{N}$:

$$\widehat{N} = \frac{\nabla g}{\|\nabla g\|} \quad \text{or} \quad \widehat{N} = -\frac{\nabla g}{\|\nabla g\|}.$$

For example, for the ellipsoid

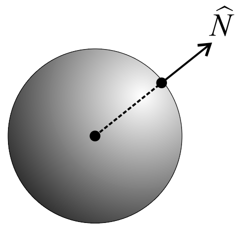
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

we may take

$$\widehat{N} = \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$

For the sphere $S(0, a)$ we have in particular either

$$\widehat{N} = \frac{(x, y, z)}{a} \quad \text{or} \quad \widehat{N} = -\frac{(x, y, z)}{a}.$$



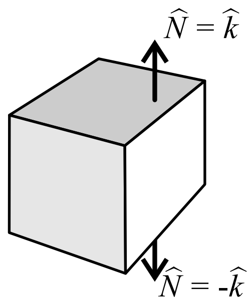
REMARK. An orientable surface is also said to be *two-sided.* The reason for this is that the continuous normal vector $\widehat{N}$ serves to define a direction of "up" at points of M. Thus at points of M there is a definite sense of *two sides* of M, an "up" side and a "down" side. A Möbius strip for example is *one-sided*, which may be demonstrated by drawing a curve along the "equator" of M with a pencil.

EXTENSION. Frequently we shall need to analyze a surface $M \subset \mathbb{R}^3$ which is not actually orientable in the above sense, but is "close enough." The surface may consist of finitely many surfaces with the proper orientability. A few examples should suffice for a good explanation.

The surface of a cube. If the cube is $[-1, 1] \times [-1, 1] \times [-1, 1]$, then the surface consists of the six squares making up the boundary of the solid cube. A typical face is

$$\{1\} \times [-1, 1] \times [-1, 1] = \{(1, y, z) \mid -1 \le y \le 1, -1 \le z \le 1\}.$$

Now we may choose $\widehat{N}$ at each point except those points on the edges to point (say) outward. Thus for the above face we have $\widehat{N} = (1, 0, 0) = \hat{\imath}$. On the opposite face $\{-1\} \times [-1, 1] \times [-1, 1]$ we would have $\widehat{N} = -\hat{\imath}$. We thus regard this surface as orientable.

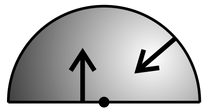


The boundary of a hemiball. For instance consider the hemiball

$$x^2 + y^2 + z^2 \le a^2, \quad z \ge 0.$$

Then the surface we have in mind consists of the hemisphere

$$x^2 + y^2 + z^2 = a^2, \quad z \ge 0,$$



together with the disk

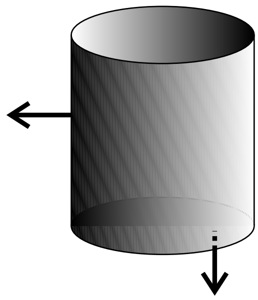
$$x^2 + y^2 \le a^2, \quad z = 0.$$

If we choose the inward normal vector, then we have

$$\widehat{N} = \frac{(-x, -y, -z)}{a} \quad \text{on the hemisphere,}$$
$$\widehat{N} = \hat{k} \quad \text{on the disk.}$$

A cylindrical can. Consider the surface for which one part is given by

$$x^2 + y^2 = a^2, \quad 0 \le z \le h,$$



and the other part by

$$x^2 + y^2 \le a^2, \quad z = 0.$$

Then we might choose an "outer" unit normal vector

$$\begin{cases} \widehat{N} & = \frac{(x,y,0)}{a} \quad \text{for } \ 0 < z \le h, \\ \widehat{N} & = -\hat{k} \quad \text{for } \ z = 0. \end{cases}$$

PROBLEM 13–1. The following surface is orientable. It consists of the union of the cylinder

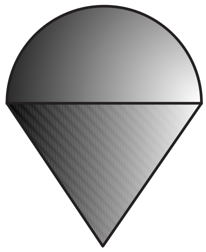
$$x^2 + y^2 = a^2, \quad 0 \le z \le h;$$

and the hemisphere

$$x^2 + y^2 + (z - h)^2 = a^2, \quad h \le z \le h + a.$$

Draw a sketch of it and write down expressions for an "outer" unit normal vector.

PROBLEM 13–2. Give formulas for an "ice cream cone" surface, consisting of a right circular cone topped off with a hemisphere. Then give formulas for the 'outer" unit normal vector.



All of the surfaces we shall be considering will be *connected*. Each will be piecewise C^1 and any two points on M can be joined by a piecewise C^1 curve lying in M. Each will be orientable as well and we shall be faced with just two choices for $\widehat{N}$. This leads to one more

DEFINITION. An orientable surface M is said to be *oriented* if a definite choice has been made of a continuous unit normal vector $\widehat{N}$ for M.

There is actually a touch of vagueness in this definition in that $\widehat{N}$ may not be continuous for a piecewise C^1 surface, and certainly may fail to exist at various points. We have escaped trouble in our examples thanks to an intuitive concept of "outer" or "inner." The full resolution of this ambiguity will be given in Section C and then again in Section F in our discussion of Stokes' theorem. But for the moment we are content to live with this ambiguity.

B. **The boundary of a surface**

This is the second feature of a surface that we need to understand. Consider a surface $M \subset \mathbb{R}^3$ and assume it's a closed set. We want to define its boundary.

To do this we cannot revert to the definition of bd$M$ given in Section 10A. For according to that definition bd$M = M$. The reason is that M has no interior points, since interior points have to do with open balls in $\mathbb{R}^3$.

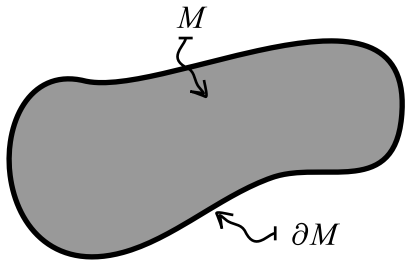
Nevertheless it is clear that we would like some concept of the boundary of M. It's easy to make it precise. We say that $p \in M$ is an interior point if there is a "disk-like' neighborhood of p which lies in M. Otherwise, p is a boundary point of M.

Another way of thinking of this concept is to imagine M as being the "universe," and dwellers in this universe have their own two-dimensional idea of interior point, and don't even know about the ambient $\mathbb{R}^3$. In other words, they think of *intrinsic* interior points of M.

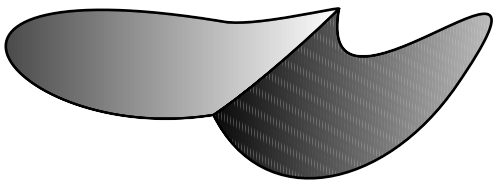
NOTATION. The set of boundary points of M will be denoted

$$\partial M.$$

Here's a typical sketch:



In another typical situation we'll have a sort of edge in M where $\widehat{N}$ is undefined:



The points in this edge are not in ∂M, as they have a "disk-like" neighborhood in M, even though the disk is bent.

EXAMPLES from the preceding section:

The surface of a cube. $\partial M = \emptyset$.

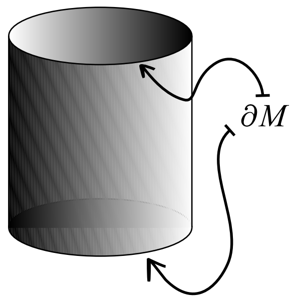
The boundary of a hemiball. $\partial M = \emptyset$.

A cylindrical can. Here ∂M consists of the top circle,

$$x^2 + y^2 = a^2, \quad z = h.$$

Another typical example is a cylinder "open at both ends":

$$x^2 + y^2 = a^2,$$
$$0 \le z \le h.$$



Here ∂M consists of the two circles

$$x^2 + y^2 = a^2, \quad z = 0 \quad \text{and} \quad x^2 + y^2 = a^2, \quad z = h.$$

DEFINITION. A surface M is *closed* if $\partial M = \emptyset$.

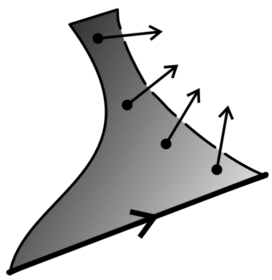
Again, this definition conflicts with our use of the same word in Section 10A. Unfortunately, this terminology has become standard. So you must be careful if someone utters the phrase "closed surface." Be sure you understand what is meant. A better term would be "surface without boundary."

Typically, if M is equal to bd$D$ for some set $D \subset \mathbb{R}^3$, then M is a closed surface.

Notice that ∂M may consist of several disjoint arcs.

C. **Inherited orientation**

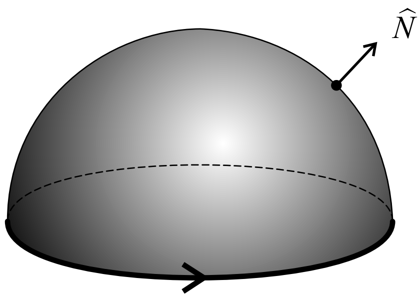
The two concepts of orientation and boundary have an important relationship. Namely, suppose the *oriented* surface M has a nonempty boundary ∂M. Consider an arc belonging to ∂M. Then we can assign a direction to ∂M by saying that if we walk along ∂M with our heads "up," then we see M at our left sides. Of course "up" refers to the chosen unit normal vector $\widehat{N}$.



We describe this by saying that ∂M ***inherits*** its orientation from M.

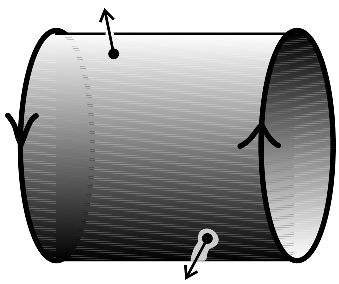
Notice that this is in complete agreement with our statement of Green's theorem. There bd$R$ is given the direction which keeps R on the left, if we suppose a third z direction pointing "up" from the $x - y$ plane with its usual orientation.

EXAMPLE. Hemisphere.



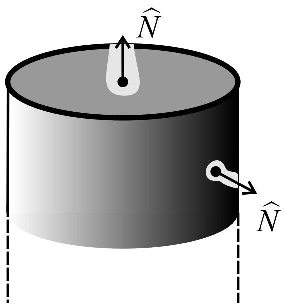
EXAMPLE. Cylinder open at both ends.

This example is extremely typical, and is quite easy, but very important to understand!



It goes without saying that if $\partial M = \emptyset$, then we need not worry about an inherited orientation.

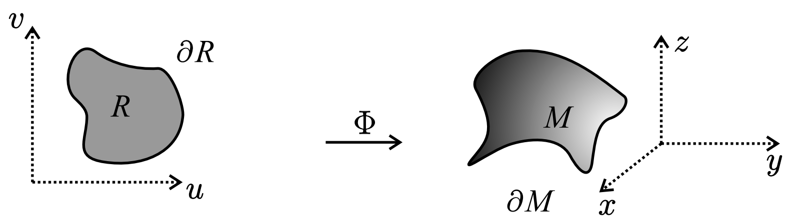
Now we can easily explain the orientation of piecewise C^1 surfaces. Each smooth piece needs to be oriented in such a way that the induced orientations given to any arc in ∂M which is in the boundary of each piece are *opposite*. A nice example is a cylinder with a top disk:



The displayed unit normal vectors $\widehat{N}$ give opposite orientations to the circular arc the two parts of the surface have in common.

D. **The basic calculation**

Now we are ready to go! We start with a small piece of an oriented surface, and we actually assume it's of class C^2. Call it $M \subset \mathbb{R}^3$. This surface is to be considered to be parameterized in the usual way. Let us call the parameters u, v, so we have a parameter mapping Φ from a region $R \subset \mathbb{R}^2$ onto M.



We know that the partial derivatives Φ_u, Φ_v give a basis for the tangent space to M at any point, and thus their cross product $\Phi_u \times \Phi_v$ is a nonzero vector normal to M. As M is oriented we are given a unit normal vector $\widehat{N}$ at each point of M. We now want to make sure that $\Phi_u \times \Phi_v$ is a *positive* scalar multiple of M. This can be achieved by the device of interchanging u and v if necessary. Thus we have

$$\widehat{N} = \frac{\Phi_u \times \Phi_v}{\mathcal{J}},$$

where the denominator $\mathcal{J}$ is simply the norm

$$\mathcal{J} = \|\Phi_u \times \Phi_v\|.$$

From Section 11B we know that area integration on M comes from $\mathcal{J}$ as the Jacobian factor:

$$d\text{area} = \mathcal{J}\, du dv.$$

Next, we make sure that we represent the parameter $u - v$ space as a right-handed coordinate system, as shown in the figure. Then we make the all-important observation that the positive direction of bd$R$ in the parameter space corresponds to the inherited orientation of ∂M in $\mathbb{R}^3$. You should check this for yourself:

PROBLEM 13–3. Prove the statement just made about the orientation.

Now we are ready for the computation. The goal we have in mind is to rewrite a general line integral of the form

$$\int_{\partial M} F \bullet d\vec{x}$$

as a surface integral of the form

$$\iint\limits_{M} (???) d\text{area}.$$

(We don't yet know what the integrand will be.) In doing this we have to integrate along ∂M in the direction inherited from $\widehat{N}$. We need a parameter for describing ∂M, and we'll just use a convenient parameter t for describing bd$R$. That is, bd$R$ may be described by functions $u = u(t)$, $v = v(t)$, and then ∂M is described by the three coordinates

$$\begin{aligned} x &= x(u(t), v(t)), \\ y &= y(u(t), v(t)), \\ z &= z(u(t), v(t)). \end{aligned}$$

Here we abuse the notation by thinking of Φ as described by three functions $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. But we'll completely suppress t in our calculation. You'll never need to see it.

Please notice that we write our surface integrals in the present chapter with two integral signs, as the manifolds we consider are always two-dimensional ones (and their one-dimensional boundaries).

Finally, we shall calculate just the particular special case of our line integrals,

$$\int_{\partial M} f dx.$$

In other words, $F = (f, 0, 0)$. We'll then easily get the other two cases by cycling through the indices.

Here we go! We have

$$\begin{aligned}
\int_{\partial M} f\,dx &\overset{\text{chain rule}}{=} \int_{\text{bd}R} f(x_u du + x_v dv) \\
&= \int_{\text{bd}R} (fx_u)du + (fx_v)dv \\
&\overset{\text{Green}}{=} \iint\limits_R [(fx_v)_u - (fx_u)_v]dudv \\
&= \iint\limits_R [f_u x_v + \underbrace{fx_{vu}} - f_v x_u - \underbrace{fx_{uv}}_{\text{CANCEL}}]dudv \\
&= \iint\limits_R [f_u x_v - f_v x_u]dudv \\
&\overset{\text{chain rule}}{=} \iint\limits_R [(\underline{f_x x_u} + f_y y_u + f_z z_u)x_v - (\underline{f_x x_v} + f_y y_v + f_z z_v)x_u]dudv \quad \text{CANCEL} \\
&= \iint\limits_R [f_y(y_u x_v - y_v x_u) + f_z(z_u x_v - z_v x_u)]dudv.
\end{aligned}$$

Now we are ready to incorporate the normal vector $\widehat{N}$ into this equation. By definition of the cross product we have

$$\begin{aligned}
\widehat{N} &= \mathcal{J}^{-1}\Phi_u \times \Phi_v \\
&= \mathcal{J}^{-1} \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix} \\
&= \mathcal{J}^{-1}(y_u z_v - y_v z_u, \quad z_u x_v - z_v x_u, \quad x_u y_v - x_v y_u) \\
&= (N_1, N_2, N_3).
\end{aligned}$$

Thus we recognize that

$$\int_{\partial M} f\,dx = \iint\limits_R (-f_y N_3 + f_z N_2)\mathcal{J}dudv.$$

Of course, $\mathcal{J}dudv = d$area. Thus if we write the integrand in the form of a dot product, we have

$$\int_{\partial M} f\,dx = \iint\limits_M (0, f_z, -f_y) \bullet \widehat{N} d\text{area}.$$

This is the end result of our calculation. The parameters have disappeared and everything is in terms of the surface M.

It is truly wonderful that just knowing the *flat* Green's theorem has led to this *curved* version by just routine manipulations of the definitions. Almost no thought was required!

"I cast it into the fire, and there came out this calf"

−Aaron, Exodus 32[24]

E. **The basic theorem, curl**

We now very quickly extend the result we have just proved by cycling through the coordinates $x \to y \to z \to x$. Thus we must have

$$\int_{\partial M} g dy = \iint_M (-g_z, 0, g_x) \bullet \widehat{N} d\text{area}$$

and

$$\int_{\partial M} h dz = \iint_M (h_y, -h_x, 0) \bullet \widehat{N} d\text{area}.$$

We then add our three basic line integrals to obtain

$$\int_{\partial M} f dx + g dy + h dz = \iint_M (h_y - g_z, f_z - h_x, g_x - f_y) \bullet \widehat{N} d\text{area}.$$

We have come to a point where we need to make a fascinating

DEFINITION. Given a vector field $F = (f, g, h)$ of class C^1 on $\mathbb{R}^3$, the *curl* of F is the vector field

$$\text{curl} F = (h_y - g_z, f_z - h_x, g_x - f_y).$$

Incidentally, most languages other than English use the word *rotation* in place of curl, and write rot$F$ for the vector field. On the next page you will find examples that have been photocopied from calculus texts in Russian, German, French, and Italian.

This vector field curl$F$ is quite amazing. You should keep in mind how very naturally it appeared in our calculations. It isn't something we had to be clever to invent; it simply arose in the calculations.

There's a wonderful mnemonic for the curl. Recall the mnemonic for the cross product of two vectors,

$$a \times b \ = \ \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

$$\mathrm{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}$$

Введем кроме того в рассмотрение вектор, составляющие которого равны разностям, стоящим под знаком двойного интеграла. Вектор этот, образующий новое векторное поле, называется вихрем поля **A** и обозначается символом rot **A** или curl **A**), так что

$$\mathrm{rot}_x\,\mathbf{A}=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z};\quad \mathrm{rot}_y\,\mathbf{A}=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x};\quad \mathrm{rot}_z\,\mathbf{A}=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}. \quad (40)$$

Формулу (39) при этом можно переписать так:

$$\int_{(l)} A_s\,ds=\iint_{(S)}[\mathrm{rot}_x\mathbf{A}\cos(n,X)+\mathrm{rot}_y\mathbf{A}\cos(n,Y)+\mathrm{rot}_z\mathbf{A}\cos(n,Z)]\,dS$$

или

$$\int_{(l)} A_s\,ds=\iint_{(S)}\mathrm{rot}_n\,\mathbf{A}\,dS, \quad (41)$$

Wir führen in die Betrachtung außerdem den Vektor ein, dessen Komponenten gleich den in dem Doppelintegral stehenden Differenzen sind. Dieser Vektor, der ein neues Vektorfeld bildet, heißt *Rotation des Feldes* $\mathfrak{a}$ und wird mit dem Symbol rot $\mathfrak{a}$ oder curl $\mathfrak{a}$ bezeichnet, so daß

$$\mathrm{rot}_x\,\mathfrak{a}=\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z};\quad \mathrm{rot}_y\,\mathfrak{a}=\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x};\quad \mathrm{rot}_z\,\mathfrak{a}=\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y} \quad (40)$$

gilt.

Die Formel (39) läßt sich damit umschreiben in

$$\int_{(l)} a_s\,ds=\iint_{(S)}[\mathrm{rot}_x\mathfrak{a}\cos(\boldsymbol{n},x)+\mathrm{rot}_y\mathfrak{a}\cos(\boldsymbol{n},y)+\mathrm{rot}_z\mathfrak{a}\cos(\boldsymbol{n},z)]\,dS$$

oder

$$\int_{(l)} a_s\,ds=\iint_{(S)}\mathrm{rot}_n\,\mathfrak{a}\,dS. \quad (41)$$

2° *Rotationnel d'un champ de vecteurs dans l'espace* E^3. — Dans l'espace $E^3\langle x,y,z\rangle$, soit $\mathbf{V}(m)$ un champ de vecteurs, défini pour $m=\langle x,y,z\rangle$ appartenant à un domaine D, de composantes P, Q, R (x,y,z); la formule de Stokes (3.9) peut s'écrire, en changeant les notations, et en considérant une courbe Γ frontière d'une surface Δ.

$$\int_{\Gamma_-}\mathbf{V}(\mathbf{m})\,dm=\mathrm{flux}_{\Delta+}(\mathbf{rot}\,\mathbf{V}),$$

où le côté Δ_+ est induit par le choix du sens de parcours sur Γ, et où l'on a désigné par **rot V** (lire *rotationnel* de **V**), le vecteur de composantes

$$\mathbf{rot}\,\mathbf{V}:\quad \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\quad \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\quad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}.$$

e, sommando queste e la (12) a membro a membro, si perviene alla formula di STOKES:

$$(13)\qquad \int_{\sigma}(X\,dx+Y\,dy+Z\,dz)=\iint_S\left[\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)dy\,dz+\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)dz\,dx+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)dx\,dy\right].$$

Se X, Y, Z s'interpretano come componenti di un vettore I, il vettore di componenti $Z'_y-Y'_z$, $X'_z-Z'_x$, $Y'_x-X'_y$ si chiama *rotazione di I*, rot I, mentre la quantità $\int_\sigma (X\,dx+Y\,dy+Z\,dz)=\int_\sigma I\times dP$ è la *circuitazione* di I lungo σ; e la (13) si scrive in forma vettoriale:

$$(14)\qquad \int_S \mathrm{rot}\, I\times n\,d\sigma=\int_\sigma I\times dP,$$

from Section 7A. If we replace the second row with the "vector"

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

then we obtain

$$\text{curl}F = \nabla \times F = \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

The meaning of this "determinant" is this: expand along the first row and regard the entries of the second row as "operating" on the functions in the third row. For example, the second component of $\nabla \times F$ is

$$-\det \begin{pmatrix} \partial/\partial x & \partial/\partial z \\ F_1 & F_3 \end{pmatrix} = -(\partial/\partial x)F_3 + (\partial/\partial z)F_1$$

$$\overset{\text{that is}}{=} -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}.$$

PROBLEM 13–4. Suppose that the surface M is presented as a graph. Specifically, assume M is given as the set

$$\{(x, y, \varphi(x, y)) \mid (x, y) \in R\},$$

where R is a region in the $x - y$ plane and φ is a C^2 function. Start from the beginning and derive Stokes' theorem for this special case. Just follow the outline given above, using the "upward" normal $\widehat{N}$ with $N_3 > 0$. The cases

$$\int_{\partial M} f dx \quad \text{and} \quad \int_{\partial M} g dy$$

are quite similar, whereas the term

$$\int_{\partial M} h dz$$

requires a slightly different treatment.

In terms of curl we can now write Stokes' theorem in the form

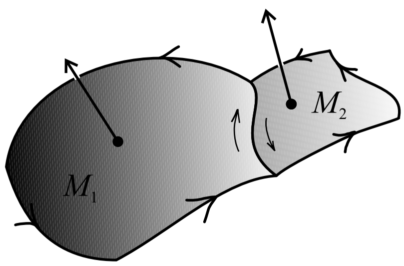
$$\int_{\partial M} F \bullet d\vec{x} = \iint_M \text{curl}F \bullet \widehat{N} d\text{area}.$$

This is our basic version of Stokes' theorem. Now we show how to extend it.

F. **Stokes' theorem**

We have proved the result in the preceding section under the restrictive hypothesis that M is presented in terms of a single parametrization. We now go through the same exercises we used in Section 12C to extend Green's theorem.

The fundamental observation is in fact the same we used for Green. If two pieces of M meet along a seam and Stokes is applied to each, the line integrals along the seam cancel each other because of our assumption on the orientation of M and the inherited orientation of ∂M. Here's a sketch:



We then can finally present our theorem as follows:

STOKES' THEOREM. *Assume M is a piecewise C^2 bounded oriented surface in $\mathbb{R}^3$ whose boundary ∂M has the inherited orientation. Assume F is a C^1 vector field defined on M. Then*

$$\int_{\partial M} F \bullet d\vec{x} = \iint_M curlF \bullet \widehat{N} darea.$$

In summary, Stokes' theorem may be regarded precisely as a *curved version of Green's theorem in the plane.*

Important special case: *In the above theorem, if M is a closed surface ($\partial M = \emptyset$), then*

$$\iint_M \mathrm{curl}F \bullet \widehat{N} d\mathrm{area} = 0.$$

EXAMPLE. Let M be the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. Let $F = (x, x, y)$. Then

$$\text{curl}F = (1, 0, 1).$$

Choose the orientation given by $\widehat{N} = \frac{(x,y,z)}{a}$. Then

$$\begin{aligned}
\iint\limits_M \text{curl}F \bullet \widehat{N} d\text{area} &= \iint\limits_M \frac{x+z}{a} d\text{area} \\
&\overset{\text{symmetry}}{=} \frac{1}{a} \iint\limits_M z d\text{area} \\
&\overset{\text{sph. coords}}{=} \frac{1}{a} \int_0^{2\pi} \int_0^{\pi/2} a \cos\varphi \; a^2 \sin\varphi d\varphi d\theta \\
&= a^2 \cdot 2\pi \cdot \frac{1}{2} = \pi a^2.
\end{aligned}$$

And

$$\begin{aligned}
\int_{\partial M} F \bullet d\vec{x} &= \int_{\partial M} x dx + x dy + y dz \\
&= 0 + \int_{\partial M} x dy + 0 \\
&= \int_0^{2\pi} a \cos\theta (a \cos\theta d\theta) = \pi a^2.
\end{aligned}$$

Another method that can be used here is to use the larger (closed) surface M' consisting of the hemisphere M and the disk $x^2 + y^2 \leq a^2$, $z = 0$. The orientation of M' requires that at points of the disk $\widehat{N} = -\hat{k}$. Stokes' theorem gives

$$\iint\limits_{M'} \text{curl}F \bullet \widehat{N} d\text{area} = 0.$$

Thus

$$\begin{aligned}\iint_M \text{curl}F \bullet \widehat{N} d\text{area} &= -\iint_{\text{disk}} \text{curl}F \bullet \widehat{N} d\text{area} \\ &= -\iint_{\text{disk}} (1,0,1) \bullet (0,0,-1) d\text{area} \\ &= \iint_{\text{disk}} d\text{area} \\ &= \text{area of disk} \\ &= \pi a^2.\end{aligned}$$

How nice! We didn't really have to compute an integral this time.

PROBLEM 13–5. Let $\hat{u} \in \mathbb{R}^3$ be a unit vector, and let $-1 < a < 1$ be fixed. Let γ be the circle defined by

$$\begin{aligned}\|(x,y,z)\| &= 1, \\ (x,y,z) \bullet \hat{u} &= a.\end{aligned}$$

Give γ the counterclockwise orientation as seen by a viewer located at the point $10\hat{u}$. Compute the line integral

$$\int_\gamma x dy.$$

PROBLEM 13–6. Let m be a fixed real number, and let γ be the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = mx$. Assume the curve has the counterclockwise orientation as viewed from $(0,0,r)$ for large positive r. Compute directly the line integral

$$\int_\gamma y dz.$$

Also compute the same line integral using Stokes' theorem.

(ANSWER: $-\pi m^3/4$.)

PROBLEM 13–7. Repeat the preceding problem but with a plane of the more general form $z = m_1 x + m_2 y$.

PROBLEM 13–8. Let M be the surface in $\mathbb{R}^3$ which is the portion of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the cylinder $x^2 + y^2 \le y$ and for which $z \ge 0$. Choose the orientation given by the unit vector $(-x, -y, -z)$. Give ∂M the inherited orientation. Calculate the six line integrals

$$\int_{\partial M} xdy \quad \text{and} \quad \int_{\partial M} ydx;$$
$$\int_{\partial M} ydz \quad \text{and} \quad \int_{\partial M} zdy;$$
$$\int_{\partial M} zdx \quad \text{and} \quad \int_{\partial M} xdz.$$

(ANSWERS: include the numbers 0, 2/3, $\pi/4$.)

PROBLEM 13–9. Let γ be the ellipse which is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = ax + by$, and give γ the counterclockwise orientation as viewed from a distant point on the positive z-axis. Calculate the line integral

$$\int_\gamma xyzdz$$

directly, and also by using Stokes' theorem.

G. **What *is* curl?**

We pause to consider some significant examples, and then to give an interpretation of curl that is completely geometric.

1. CENTRAL SYMMETRIC FIELD. A vector field is said to be *central* if all values of it point toward a fixed point. We may conveniently take this fixed point to be the origin, so a vector field on $\mathbb{R}^n$ is central if and only if it is given by an expression of the form

$$F(x) = \varphi(x)x \quad \text{for} \quad x \in \mathbb{R}^n,$$

where of course φ is a real-valued function. We then say F is also *symmetric* if $\varphi(x)$ depends only on $\|x\|$. For the case of $\mathbb{R}^3$ this becomes

$$F(x, y, z) = g(r)(x, y, z),$$

where of course $r = \|(x,y,z)\|$ is the usual spherical coordinate. Then for example the first component of curl$F$ equals

$$\begin{aligned}\frac{\partial}{\partial y}(g(r)z) - \frac{\partial}{\partial z}(g(r)y) &= g'(r)\frac{yz}{r} - g'(r)\frac{zy}{r} \\ &= 0.\end{aligned}$$

Thus curl$F = 0$.

2. CONSERVATIVE FIELD. Our definition from Section 12 states that a vector field on $\mathbb{R}^3$ is conservative if there exists a potential function f such that $F = \nabla f$. Assuming that f is of class C^2, we conclude that curl$F = 0$. For instance, the second coordinate of curl$F$ is

$$\begin{aligned}\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z} \\ &= 0.\end{aligned}$$

Thus we have the interesting result, that always

$$\text{curl grad} f = 0.$$

Or in terms of the "del" notation,

$$\boxed{\nabla \times \nabla f = 0.}$$

3. REMARK. Actually, Example 1 is a special case of Example 2, as *every* central symmetric field is conservative. This is even true for $\mathbb{R}^n$. For suppose

$$F(x) = \varphi(r)x$$

is given on $\mathbb{R}^n$, and we want to find a potential function for F. We would certainly expect this potential to be spherically symmetric itself, so we look for a function of the form $f(r)$. We thus want

$$\nabla(f(r)) = \varphi(r)x;$$

that is,

$$f'(r)\frac{x}{r} = \varphi(r)x;$$

we thus simply need to integrate the equation

$$f'(r) = r\varphi(r)$$

in order to find f.

4. ZERO CURL. We now can explain the strange terminology of Section 12E. There we said that a vector field F on $\mathbb{R}^n$ with

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

has *zero curl*. In case $n = 3$ this is exactly the condition that curl$F = 0$, so by analogy we say the same for general n. It's just that for $n \neq 3$ we don't actually have a vector field we call curl$F$.

5. PLANAR FIELDS. Suppose a vector field on $\mathbb{R}^3$ has the special form

$$F(x, y, z) = (F_1(x, y), F_2(x, y), 0),$$

so that F is parallel to the plane $z = 0$ and also is independent of z. Then

$$\text{curl}F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}.$$

6. EXAMPLES OF PLANAR FIELDS. Our first example comes from the famous $\nabla\theta$ on $\mathbb{R}^2$, so that

$$F = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right).$$

Then curl$F = 0$. Two more significant examples are

$$\text{curl}(-y, x, 0) = 2\hat{k},$$
$$\text{curl}(y, 0, 0) = -\hat{k}.$$

PROBLEM 13–10. Let $a \in \mathbb{R}^3$ be an arbitrary fixed vector. Show that

$$\nabla \times (a \times \vec{x}) = 2a.$$

(Here $\vec{x}$ stands for (x, y, z).)

PROBLEM 13–11. The vector field $a \times \vec{x}$ of the preceding problem is a special case of a general linear vector field. Such a field can be written $F(\vec{x}) = A\vec{x}$, where A is a real 3×3 matrix and $\vec{x}$ is written as a column vector. Show that

$$\nabla \times F = (a_{32} - a_{23},\ a_{13} - a_{31},\ a_{21} - a_{12}).$$

PROBLEM 13–12. Here is a problem from *American Mathematical Monthly*, Volume 109, Number 7, August-September, 2002, proposed by Victor Alexandrov, Sobolev Institute of Mathematics, Novosibirsk, Russia:

Let M be a surface contained in the unit sphere in $\mathbb{R}^3$. Use the outer unit normal vector $\widehat{N}$ for the sphere. In addition, let $\hat{n}$ denote the unit vector at points of ∂M which is tangent to the unit sphere and is orthogonal to ∂M. (In particular, $\hat{n} \bullet \widehat{N} = 0$.) Prove that

$$\int\limits_{\partial M} \hat{n} ds + 2 \iint\limits_{M} \widehat{N} d\text{area} = 0.$$

Here ds denotes arc length and the integrals with vector integrands are to be interpreted in the sense of integrating the components and then combining the results into vectors. (HINT: apply Stokes to the vector field $F = a \times \vec{x}$ for any fixed vector a.)

PROBLEM 13–13. Let C denote the unit circle

$$\{(x, y, 0) \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^3$. Define the "function" g on $\mathbb{R}^3 - C$ by

$$g(x, y, z) = \arctan \frac{z}{x^2 + y^2 - 1}.$$

Of course, g is not completely well defined.

a. Prove that the vector field $F = \nabla g$ *is* a well defined vector field on $\mathbb{R}^3 - C$, and that it is C^∞.

b. Prove that F is irrotational.

c. Prove that F is not conservative by showing that there is a loop in $\mathbb{R}^3 - C$ along which the line integral of F is not zero.

d. Calculate F explicitly, and show that on the specific loop γ

$$y = 0,\ z^2 = 2x^2 - x^4 \qquad (x \geq 0),$$

the line integral of F equals

$$\int_\gamma -2xzdx + (x^2 - 1)dz.$$

Calculate this integral directly and show it equals 2π.

GEOMETRY. We now present a geometric interpretation of curl which provides intuition for the above examples and others. First, we need to understand line integrals in a certain geometric way. Suppose that F is a vector field on $\mathbb{R}^n$ and γ is a curve in $\mathbb{R}^n$, say $\gamma = \gamma(t)$, $a \leq t \leq b$. Then the dot product

$$F(\gamma(t)) \bullet \gamma'(t)$$

represents the component of F in the direction tangent to the curve, multiplied by the speed of the curve. Thus in a very significant sense the integral

$$\int_a^b F(\gamma(t)) \bullet \gamma'(t)dt$$

represents the total net component of F tangent to the curve γ. In particular, if γ is a closed curve, then this integral represents the net increase of the tangential component of F around γ. We thus say that

$$\int_\gamma F \bullet d\vec{x} \;=\; \text{the net } \textit{circulation} \text{ of } F \text{ around } \gamma.$$

In case F represents a force, then we might think of the circulation as the net *work* done by F in going around γ. In case F represents the velocity of a fluid, then it would be the net flow of the fluid.

In particular, a vector field is conservative $\Longleftrightarrow$ it has zero circulation around every closed curve.

PROBLEM 13–14. Let F be a vector field in $\mathbb{R}^2$ which is given by

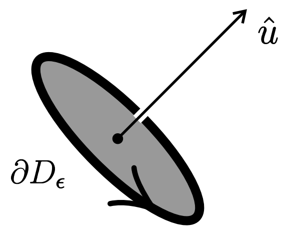
$$F \;=\; r^\alpha(-y, x). \qquad (r = \sqrt{x^2+y^2})$$

Find the net circulation of F around the counterclockwise circle $x^2+y^2=a^2$. For which value of α is the result independent of the radius a?

Stokes' theorem thus asserts that in the case of $\mathbb{R}^3$ the net circulation of F around ∂M may be measured by calculating the *surface* integral over M of the normal component of the curl of F (with proper orientation).

This idea may be presented infinitesimally in such a way as to give an entirely different way of defining curl. To see this, suppose F is a vector field on $\mathbb{R}^2$ which is defined in some neighborhood of a fixed point p_0. We shall derive a formula for curl$F(p_0)$ without using any coordinate system and even without using any partial derivatives of any components of F!

In order to know curl$F(p_0)$, it suffices to know the number curl$F(p_0) \bullet \hat{u}$ for any unit vector $\hat{u}$. This dot product is of course the component of the vector curl$F(p_0)$ in the direction $\hat{u}$. Construct the disk D_ϵ with center p_0, radius ϵ, orthogonal to $\hat{u}$.



Use the vector $\hat{u}$ as the unit normal vector for D_ϵ, thus rendering D_ϵ an oriented surface. The bounding circle ∂D_ϵ is of course supplied with the induced orientation: it travels counterclockwise as viewed from the tip of $\hat{u}$. Then Stokes' theorem gives

$$\iint_{D_\epsilon} \text{curl} F \bullet \hat{u} d\text{area} \;=\; \iint_{\partial D_\epsilon} F \bullet d\vec{x}.$$

If ϵ is small, the left side of this equation is very close to $\text{curl}F(p_0) \bullet \hat{u}$ times the area of D_ϵ, thanks to the continuity of the integrand. Thus if we divide by $\pi\epsilon^2$ and let $\epsilon \to 0$, we obtain

$$\boxed{\text{curl}F(p_0) \bullet \hat{u} \;=\; \lim_{\epsilon \to 0} \frac{1}{\pi\epsilon^2} \int_{\partial D_\epsilon} F \bullet d\vec{x}.}$$

There is a significant way to think about the right side of this equation in terms of the idea of circulation, and this lead to the sentence

$$\boxed{\text{curl}F(p_0) \bullet \hat{u} \;=\; \text{“the counterclockwise circulation per unit area of } F \textit{ at } p_0 \text{ with respect to } \hat{u}.\text{”}}$$

MORAL. The right sides of these expressions have a definite meaning *independent of any choice of coordinate system.* Thus the same must be true of $\text{curl}F(p_0) \bullet \hat{u}$. As this is true for every choice of the unit vector $\hat{u}$, we conclude that

> the curl of a vector field on $\mathbb{R}^3$ is a geometric property of the field, depending only on the choice of the orientation of $\mathbb{R}^3$.

Notice how very naturally this description of curl has arisen. Natural as it is, it is nevertheless *stunning* in view of our initial definition in terms of the coordinate system,

$$\text{curl}(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}) \;=\; \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

All three rows of the matrix depend on the coordinate system, but the output does not!

Once again we observe the wonderful interplay between algebra and geometry! Stokes' theorem provides us with the deep geometric significance of curl, while the initial definition gives us a handy way of computing the curl of vector field.

For instance, suppose $\{\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3\}$ is a right-handed orthonormal frame for $\mathbb{R}^3$ and define coordinates t_1, t_2, t_3 by the formula

$$(x, y, z) \;=\; t_1\hat{\phi}_1 + t_2\hat{\phi}_2 + t_3\hat{\phi}_3.$$

Then we conclude immediately that

$$\text{curl}F \;=\; \det\begin{pmatrix} \hat{\phi}_1 & \hat{\phi}_2 & \hat{\phi}_3 \\ \partial/\partial t_1 & \partial/\partial t_2 & \partial/\partial t_3 \\ F\bullet\hat{\phi}_1 & F\bullet\hat{\phi}_2 & F\bullet\hat{\phi}_3 \end{pmatrix}.$$

No calculations needed!

PROBLEM 13–15. How does this formula change if $\{\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3\}$ is a *left*-handed orthonormal frame?

This is a good time to remember that we stressed a similar geometry/algebra connection back in Section 2H in our discussion of the gradient of a function. We thus arrive at *two* significant geometric insights into the "differential operator" $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, one when it acts on functions to produce conservative vector fields ∇f, and now one when it acts on vector fields to produce new vector fields $\nabla \times F$.

You should also notice why curl$F$ is also called the *rotation* of F and is sometimes written rot$F$, as we mentioned in Section E. It all has to do with the geometric description of curl in terms of the circulation, or rotation, of the vector field.

There are some straightforward calculus results that are frequently quite helpful in manipulating curl. We have seen one already: $\nabla \times \nabla f = 0$. Here is a useful product rule:

PROBLEM 13–16. Show that

$$\nabla \times (fF) \;=\; f\nabla \times F + \nabla f \times F.$$

In Chapter 15 we shall greatly extend the formula given above for curl$F$ to allow nonorthogonal frames, and in fact to allow curvilinear coordinates.

H. **Curlometer**

Imagine a fluid flowing in $\mathbb{R}^3$, and imagine the velocity vector at each point $\vec{x}$. This gives some sort of a vector field $F(\vec{x})$, which we suppose to be independent of time.

The **streamlines** of this fluid are obtained by solving the system of ordinary differential equations for the curves $\vec{x}(t)$:

$$\frac{d\vec{x}}{dt} \;=\; F(\vec{x}(t)).$$

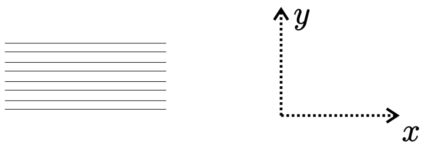
Now imagine that we want to have a mechanical device for measuring the curl of F. Imagine then a propeller free to spin, attached to a movable stick:

We might then situate the propeller at any point and the stick pointing in any direction. The speed (and direction) of rotation could then presumably describe the component of $\nabla \times F$ at that point and in that direction. This is what Stokes' theorem guarantees.

We now give three illuminating examples of this idea. Each is a planar field, so that $F = (F_1(x, y), F_2(x, y), 0)$. We then know of course that

$$\nabla \times F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}.$$

1. SHEAR FLOW. Here $F = (ay, 0, 0)$. Here's a picture of the streamlines, supposing $a > 0$.



But the lines with y large are moving faster than those with y small, so we expect that $\nabla \times F$ will be nonzero. In fact,

$$\nabla \times F = -a\hat{k}.$$

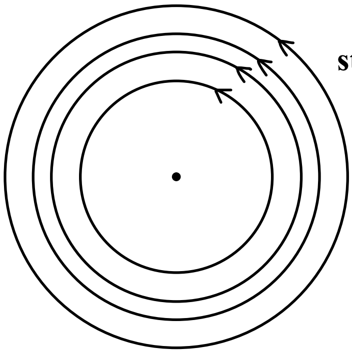
Thus our propeller experiences a clockwise rotation (supposing $a > 0$), just as our intuition expects.

2. CIRCULAR FLOW. Here $F =$ "$\nabla\theta$" $= \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$. We know that $\nabla \times F = 0$. This was mentioned in Section G6. So our propeller would tend to move in counterclockwise circles around the origin, but with no tendency to rotate as it moves.

On the other hand, consider $F = (-y, x, 0)$. The streamlines are still counterclockwise circles, but now

$$\nabla \times F = 2\hat{k},$$

which means that the propeller will spin counterclockwise as it rotates.



3. CENTRAL FLOW. Here we take $F = f(\theta)(x, y, 0)$, so that the streamlines are rays through the origin. However, the velocity may vary with θ.

PROBLEM 13–17. Show that in this example

$$\nabla \times F = -f'(\theta)\hat{k}.$$

I. Conservative fields revisited

In Section 12F we considered vector fields defined on open subsets $D \subset \mathbb{R}^n$ which had "zero curl." We noticed particularly the simple example in $\mathbb{R}^2 - \{0\}$ described as $F = \nabla\theta$. This field is irrotational but not conservative. In case D is simply connected in $\mathbb{R}^2$, we saw that irrotational fields are indeed conservative.

The same is true for simply connected open subsets of $\mathbb{R}^3$. For instance, suppose that F is a vector field of class C^1 on $\mathbb{R}^3 - \{0\}$ and suppose that $\nabla \times F = 0$. Then we can prove that F is conservative. For consider a sufficiently "nice" closed curve $\gamma \subset \mathbb{R}^3 - \{0\}$. Then we can construct an oriented surface M also contained in $\mathbb{R}^3 - \{0\}$, with $\partial M = \gamma$. Stokes' theorem then gives

$$\begin{aligned}\int_{\partial M} F \bullet d\vec{x} &= \iint\limits_M \nabla \times F \bullet \widehat{N} d\text{area} \\ &= \iint\limits_M 0 \, d\text{area} \\ &= 0.\end{aligned}$$

This verifies the validity of the second criterion in the theorem of Section 12E, and thus F is conservative.

The reason for the difference between $\mathbb{R}^2 - \{0\}$ and $\mathbb{R}^3 - \{0\}$ is clear: in $\mathbb{R}^3 - \{0\}$ there is room to maneuver to fill in a loop with a surface missing the origin.

PROBLEM 13–18. Show that the same result holds for the open set

$$\mathbb{R}^3 - \mathbb{R} \times [0, \infty) \times \{0\}.$$