

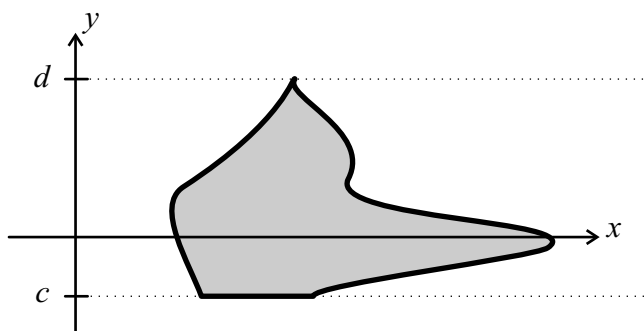
Chapter 12 Green's theorem

We are now going to begin at last to connect differentiation and integration in multivariable calculus. In addition to all our standard integration techniques, such as Fubini's theorem and the Jacobian formula for changing variables, we now add the fundamental theorem of calculus to the scene. In fact, Green's theorem may very well be regarded as a direct application of this fundamental theorem.

A. The basic theorem of Green

Consider the following type of region R contained in \mathbb{R}^2 , which we regard as the $x-y$ plane. We use the standard orientation, so that a 90° counterclockwise rotation moves the positive x -axis to the positive y -axis. Then we assume the existence of two continuous functions $a(y)$ and $b(y)$, defined for $c \leq y \leq d$, where $a(y) < b(y)$ for $c < y < d$ and

$$R = \{(x, y) \mid a(y) \leq x \leq b(y)\}.$$



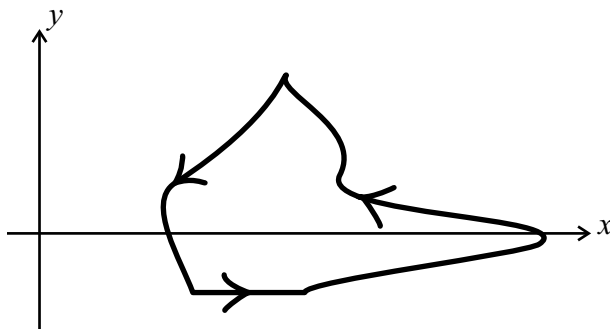
Given a function $R \xrightarrow{F} \mathbb{R}$ of class C^1 , we then compute by means of Fubini's theorem

$$\begin{aligned} \iint_R \frac{\partial F}{\partial x} dx dy &= \int_c^d \int_{a(y)}^{b(y)} \frac{\partial F}{\partial x} dx dy \\ &\stackrel{\text{FTC}}{=} \int_c^d F(x, y) \Big|_{x=a(y)}^{x=b(y)} dy \\ &= \int_c^d F(b(y), y) dy - \int_c^d F(a(y), y) dy. \end{aligned}$$

We now write the right side of this equation as

$$\int_{\text{bd}R} F dy,$$

where we mean by this notation the counterclockwise integration of F , restricted to $\text{bd}R$, with respect to y . (The actual definition is given in the formula.)



Notice that on a horizontal portion of $\text{bd}R$, y is constant and we thus interpret $dy = 0$ there.

B. Line integrals

We have now met an entirely new kind of integral, the integral along the counterclockwise $\text{bd}R$ seen above. Before proceeding further, we need to discuss this sort of oriented integral.

It will prove useful to do this in more generality, so we consider a curve γ in \mathbb{R}^n which is of class C^1 . Thus, $[a, b] \xrightarrow{\gamma} \mathbb{R}^n$ is of class C^1 . We typically denote the independent variable (the “parameter”) as t . If f is a continuous real-valued-function defined on \mathbb{R}^n , we then define the *line integral*

$$\int_{\gamma} f dx_i = \int_a^b f(\gamma(t)) \gamma'_i(t) dt.$$

Here of course $1 \leq i \leq n$. The notation is intended to be very suggestive and to lead us to the appropriate substitutions $x = \gamma(t)$, $x_i = \gamma_i(t)$, and $dx_i = \gamma'_i(t) dt$.

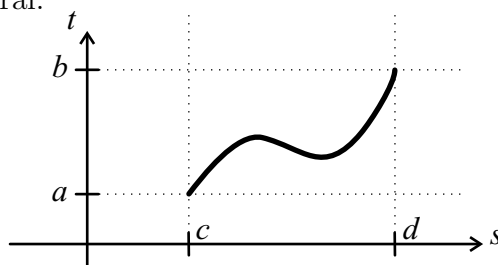
It is an easy matter to imagine some useful properties of this sort of integral, and even easier to prove them.

INDEPENDENCE OF PARAMETER CHANGE

Suppose first that $\gamma(t)$, $a \leq t \leq b$, is represented instead as $\gamma(h(s))$, $c \leq s \leq d$, where h is a C^1 function such that $h(c) = a$ and $h(d) = b$. Then using the parametrization $\gamma(h(s))$ leads to the line integral

$$\begin{aligned} \int_c^d f(\gamma(h(s))) \frac{d}{ds} \gamma_i(h(s)) ds &= \int_c^d f(\gamma(h(s))) \gamma'_i(h(s)) h'(s) ds \\ &\stackrel{t=h(s)}{=} \int_a^b f(\gamma(t)) \gamma'_i(t) dt, \end{aligned}$$

which is the original line integral.

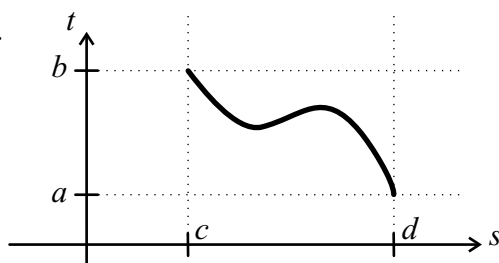


This proves the desired independence.

On the other hand, if instead $h(c) = b$ and $h(d) = a$, then we obtain

$$\int_c^d f(\gamma(h(s))) \frac{d}{ds} \gamma_i(h(s)) ds = - \int_a^b f(\gamma(t)) \gamma'_i(t) dt,$$

so we get the anticipated change of sign.



LINEARITY

This is virtually obvious from the definition:

$$\begin{aligned} \int_{\gamma} a f dx_i &= a \int_{\gamma} f dx_i \quad \text{if } a \text{ is a constant;} \\ \int_{\gamma} (f + g) dx_i &= \int_{\gamma} f dx_i + \int_{\gamma} g dx_i. \end{aligned}$$

COMBINING CURVES

Suppose we are given curves γ and δ such that the final point of γ equals the initial point of δ . Then we can think of a new curve that covers γ and then δ . This curve will be continuous but not necessarily C^1 . We denote this curve loosely as $\gamma + \delta$, and then the clear result is

$$\int_{\gamma+\delta} f dx_i = \int_{\gamma} f dx_i + \int_{\delta} f dx_i.$$

GENERALIZATION

If f_1, \dots, f_n are given functions, then we define

$$\int_{\gamma} f_1 dx_1 + \dots + f_n dx_n = \int_{\gamma} f_1 dx_1 + \dots + \int_{\gamma} f_n dx_n.$$

Another way to think of this is to arrange the functions into a vector $F = (f_1, \dots, f_n)$ and formally write

$$\begin{aligned}\vec{x} &= (x_1, \dots, x_n), \\ d\vec{x} &= (dx_1, \dots, dx_n), \\ F \bullet d\vec{x} &= f_1 dx_1 + \dots + f_n dx_n.\end{aligned}$$

Thus we have the abbreviation

$$\int_{\gamma} F \bullet d\vec{x} = \int_{\gamma} f_1 dx_1 + \dots + f_n dx_n.$$

FUNDAMENTAL THEOREM OF CALCULUS

Suppose that f is a real-valued function of class C^1 and that

$$F = \nabla f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n).$$

Then we calculate directly

$$\begin{aligned}\int_{\gamma} \nabla f \bullet d\vec{x} &= \int_a^b \nabla f(\gamma(t)) \bullet \gamma'(t) dt \\ &\stackrel{\text{chain rule}}{=} \int_a^b \frac{d}{dt} f(\gamma(t)) dt \\ &\stackrel{\text{FTC}}{=} f(\gamma(t)) \Big|_a^b \\ &= f(\gamma(b)) - f(\gamma(a)).\end{aligned}$$

Thus we have proved

$$\int_{\gamma} \nabla f \bullet d\vec{x} = f(\text{final point of } \gamma) - f(\text{initial point of } \gamma).$$

We shall say much more about this equation in Section E.

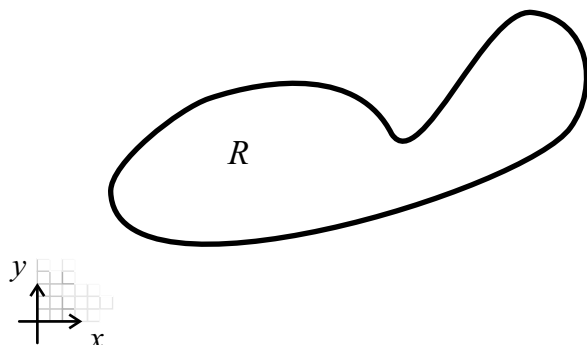
C. A general Green's theorem

We now return to the formula of Section A,

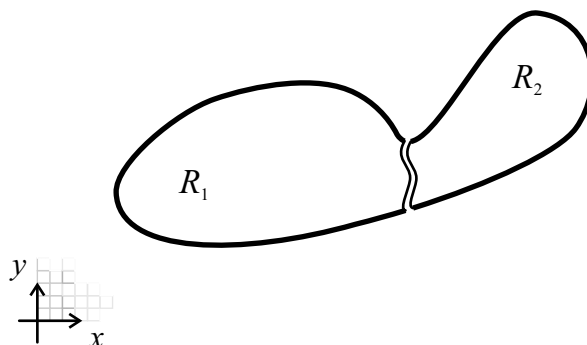
$$\iint_R \frac{\partial F}{\partial x} dx dy = \int_{\text{bd}R} F dy. \quad (*)$$

The right side is now completely understood as a line integral taken along the curve $\text{bd}R$ with its counterclockwise orientation. We presently have severe restrictions on what the region R can be, and we now show how we may easily remove many of these restrictions.

Rather than give precise definitions to delineate the allowable regions, we prefer to rely on pictures. Suppose then we have a region in \mathbb{R}^2 that looks like this:



Such a region fails to satisfy the conditions of Section A. However, the desired equation (*) still holds for it. An easy way to see this is to divide R into two regions R_1 and R_2 by means of a “cut,” in such a way that (*) is valid for each piece separately.



We thus have

$$\iint_{R_1} \frac{\partial F}{\partial x} dx dy = \int_{\text{bd}R_1} F dy,$$

$$\iint_{R_2} \frac{\partial F}{\partial x} dx dy = \int_{\text{bd}R_2} F dy.$$

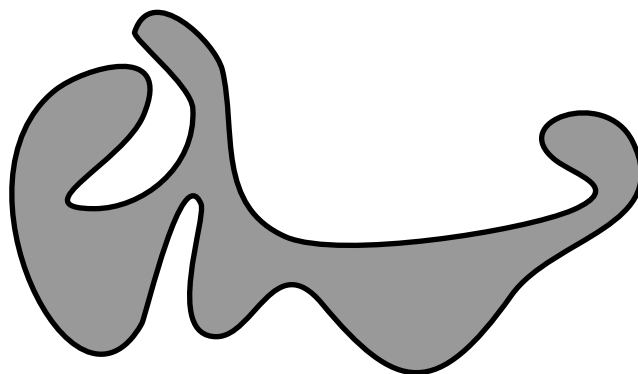
Now add these two equations. The left sides produce the double integral over R itself, and the right sides combine in an interesting way. Namely, the part of each line integral corresponding

to the cut appears in each, but with *opposite* orientations. Thus they cancel to produce just

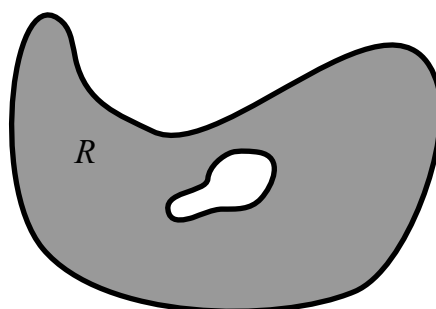
$$\int_{\text{bd}R} F dy.$$

We therefore find that (*) remains valid for this R .

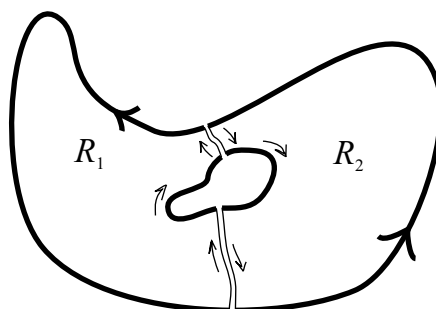
Of course, this procedure may be extended to several cuts, so that (*) remains valid for a region such as



One more generalization allows *holes* to appear in R , as for example



We again make cuts to get a couple of regions



Then we apply (*) to R_1 and R_2 and add the results, noting the cancellation of the integrations taken along the cuts. The result still is (*), but with an interesting distinction: the line integral along the inner portion of $\text{bd}R$ actually goes in the *clockwise* direction.

A convenient way of expressing this result is to say that (*) holds, where the orientation of $\text{bd}R$ is such that the interior of R is located to the *left* of the path of integration.

Finally, we close this section with the corresponding result

$$\iint_R \frac{\partial F}{\partial y} dx dy = - \int_{\text{bd}R} F dx. \quad (**)$$

The minus sign appears naturally, as we see from rewriting the basic case of Section A. Another way to think about it is to realize that the coordinate system with y first, x second, has the opposite orientation to the one we have been working with.

Most texts combine these two formulas into a single one by using different letters for the two cases, and adding. Thus,

$$\boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\text{bd}R} P dx + Q dy.}$$

D. Areas by means of Green

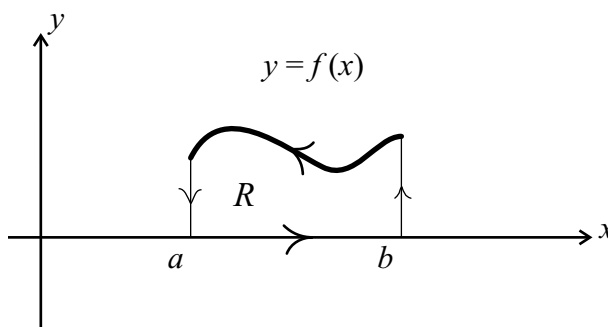
An astonishing use of Green's theorem is to calculate some rather interesting areas. These come for example by using $F = x$ in (*) in Section C to obtain

$$\text{area}(R) = \int_{\text{bd}R} x dy.$$

Likewise, using $F = y$ in (**) produces

$$\text{area}(R) = - \int_{\text{bd}R} y dx.$$

The latter equation resembles the standard beginning calculus formula for area under a graph:



The integral

$$\int_a^b f(x) dx$$

is exactly the line integral

$$-\int y dx$$

taken around $\text{bd}R$.

It is interesting that the sum of these two formulas is often more easily exploited. Or use $Q = x$, $P = -y$ in the formula at the end of Section C. The result is

$$\text{area}(R) = \frac{1}{2} \int_{\text{bd}R} x dy - y dx.$$

This is often useful in situations where x and y appear rather symmetrically.

ELLIPSE. We of course know from several investigations that the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to πab . This example gives a nice illustration of our new formula. Namely, parametrize the ellipse by $x = a \cos t$, $y = b \sin t$, where $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} x dy - y dx &= a \cos t \cdot b \cos t dt - b \sin t \cdot (-a \sin t dt) \\ &= ab(\cos^2 t + \sin^2 t) dt \\ &= ab dt. \end{aligned}$$

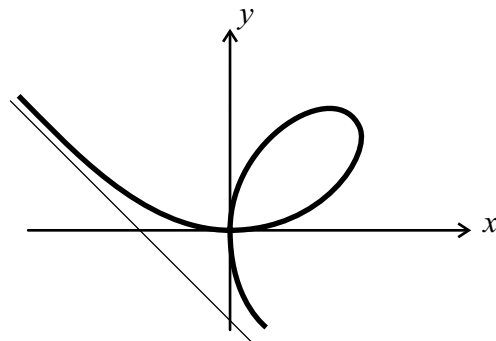
Thus the area equals

$$\frac{1}{2} \int_0^{2\pi} ab dt = \pi ab.$$

Don't miss the fact that if we had just integrated $x dy$ or $-y dx$ to find this area, the final integration would have been significantly more difficult.

FOLIUM OF DESCARTES. Here is a truly significant example, one for which our formula provides what is perhaps the easiest method for calculating a certain area. We first discussed this curve in Section 6E, where we used the implicit equation $x^3 + y^3 = 3xy$. In terms of the ratio $t = y/x$, we have the parametric presentation

$$\begin{aligned} x &= \frac{3t}{1+t^3}, \\ y &= \frac{3t^2}{1+t^3}. \end{aligned}$$



Now we find the area of the “leaf,” that region enclosed by the part of the curve in the first quadrant. Here the parameter t runs between 0 and ∞ . We calculate

$$\begin{aligned} xdy - ydx &= xd(tx) - txdx \\ &= x(tdx + xdt) - txdx \\ &= x^2dt \\ &= \frac{9t^2dt}{(1+t^3)^2} \\ &= d\left(\frac{-3}{1+t^3}\right). \end{aligned}$$

Thus

$$\begin{aligned} \text{area} &= \frac{1}{2} \int_0^\infty d\left(\frac{-3}{1+t^3}\right) \\ &= -\frac{3}{2} \frac{1}{1+t^3} \Big|_0^\infty \\ &= \frac{3}{2}. \end{aligned}$$

PROBLEM 12–1. Obtain the area of the leaf of Descartes by using just the formula

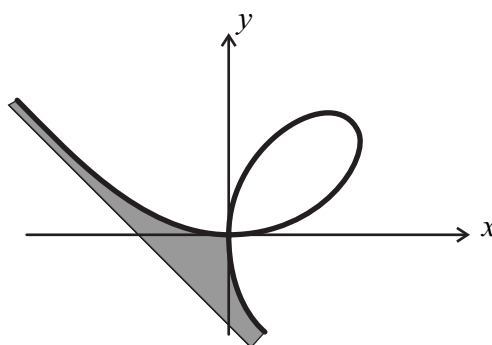
$$\text{area}(R) = - \int_{\text{bd}R} ydx.$$

PROBLEM 12–2. Show that the equations

$$\begin{aligned} \text{area}(R) &= - \int_{\text{bd}R} ydx \\ &= \int_{\text{bd}R} xdy \end{aligned}$$

can be regarded as equivalent by means of an integration by parts.

PROBLEM 12–3*. Find the area between the folium of Descartes and its asymptote $x + y = -1$, as shown in the figure.



PROBLEM 12–4. Find the area enclosed by the curve

$$x^4 + y^4 = 4xy$$

in the first quadrant.

PROBLEM 12–5. Consider the curve defined by the equation $x^4 = xy^2 + y^3$.

- Use the parameter t defined by $y = tx$ to express the curve in parametric form.
- Show that this curve has a “leaf” in the fourth quadrant.
- Calculate the area of the leaf. (ANSWER: $1/210$).

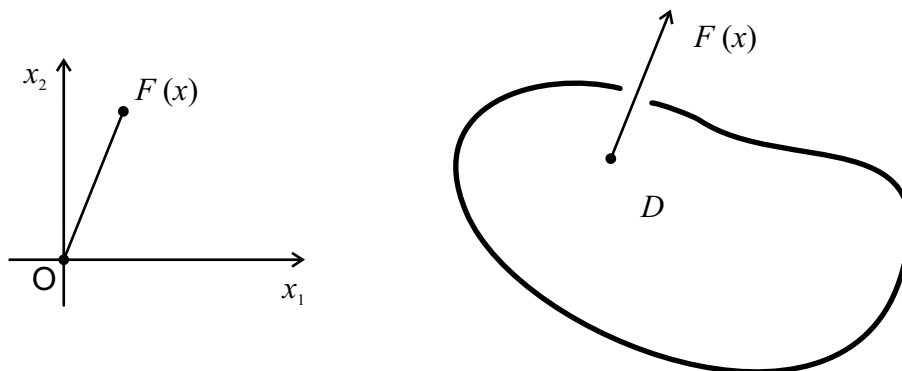
E. Conservative vector fields

Now we return to our discussion of line integrals in general, as they were introduced in Section B.

DEFINITION. Let D be an open subset of \mathbb{R}^n . A *vector field* on D is a continuous function

$$D \xrightarrow{F} \mathbb{R}^n.$$

Thus F assigns to each point $x \in D$ a point $F(x) \in \mathbb{R}^n$. We always think of $F(x)$ as a vector attached to the point x . Thus we have a picture of the following sort:



Using the components of $F = (F_1, F_2, \dots, F_n)$, we have defined line integrals of the form

$$\int_{\gamma} F_1 dx_1 + \dots + F_n dx_n.$$

In Section B we introduced the notation

$$\int_{\gamma} F \bullet d\vec{x}$$

for these line integrals.

Now we state an important basic theorem about these vector fields.

THEOREM. *Let D be an open subset of \mathbb{R}^n , and let F be a vector field defined on D . Then the following three conditions are equivalent.*

1. *The line integral $\int_{\gamma} F \bullet d\vec{x}$ is **independent of path**, in the sense that its value depends only on the initial point of γ and the final point of γ .*
2. *The line integral $\int_{\gamma} F \bullet d\vec{x} = 0$ for every **loop** in D (a loop is a curve γ whose initial point equals its final point).*
3. *There exists a function $D \xrightarrow{f} \mathbb{R}$ of class C^1 such that*

$$F = \nabla f.$$

PROOF. The proof that $1 \Rightarrow 2$ is just based on the simple observation that a loop which starts at x_0 and ends at x_0 and the constant path which just stays at x_0 are two paths with the same initial and final points. As the line integral over the constant path is zero, condition 1 implies that the line integral around the loop is zero as well.

The proof that $2 \Rightarrow 1$ is based on another simple observation: if γ_1 and γ_2 are two paths with the same initial and final points, then the path that goes along γ_1 and then backwards along γ_2 is a loop. Thus condition 2 implies

$$\int_{\gamma_1} F \bullet d\vec{x} + \int_{\gamma_2 \text{reversed}} F \bullet d\vec{x} = 0.$$

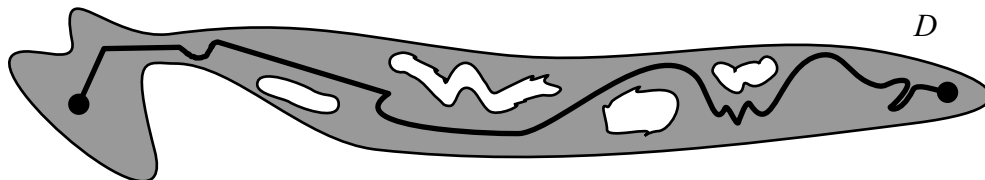
This proves condition 1.

The proof that $3 \Rightarrow 1$ is an immediate consequence of the last formula in Section B.

The really significant part of this theorem is the proof that $1 \Rightarrow 3$.

Before we introduce the crucial construction of the function f we notice that we may as well assume that our open set D is *connected*. In other words, that any two points in D can be joined by some curve lying in D . For we may work on each connected component of D and produce the required function f on each of them in turn.

In this context it is important to observe that any two points in the connected set D can in fact be joined by a “nice” curve, say one of class C^1 or one which is polygonal. For we need to be able to do line integrals along such curves.



Now our definition of f is actually forced upon us. For given the vector field F , if there is to be a function such that $F = \nabla f$, then we know from Section B that

$$f(\text{final point of } \gamma) - f(\text{initial point of } \gamma) = \int_{\gamma} F \bullet d\vec{x}.$$

Now choose an arbitrary but fixed point $b \in D$. It will serve as a “base point.” Then for any $x \in D$ we produce a curve γ in D whose initial point is b and whose final point is x . Then our “potential” function f has to satisfy

$$f(x) = f(b) + \int_{\gamma} F \bullet d\vec{x}.$$

Since b is fixed, $f(b)$ is a constant; we may as well call it 0, as adding a constant to f does not disturb our desired equation $\nabla f = F$. Thus we now *define* $D \xrightarrow{f} \mathbb{R}$ by the equation

$$f(x) = \int_{\gamma} F \bullet d\vec{x},$$

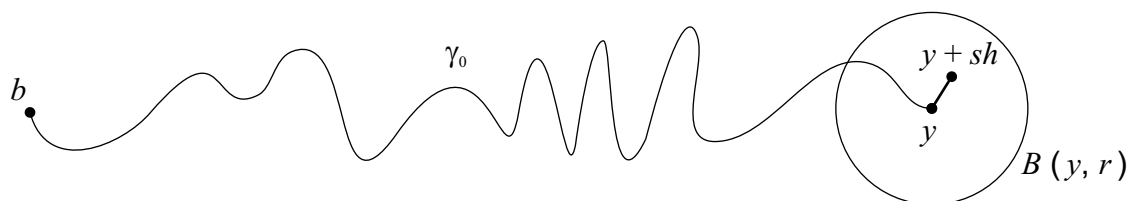
where γ is any “nice” curve in D starting at b and ending at x .

We have now in fact used the hypothesis 1, since the value given for $f(x)$ *does not depend on the choice of γ* .

Finally we must verify that $\nabla f = F$. We'll present two proofs. These proofs are indeed related, but they have distinct emphases.

FIRST PROOF. Consider any fixed point $y \in D$. Let γ_0 be a “nice” curve in D starting at b and ending at y . Also let $B(y, r)$ be an open ball contained in D . For any unit vector $\hat{h} \in \mathbb{R}^n$ and any $0 \leq s < r$ we use the curve γ_0 followed by the curve which is the line segment from y to $y + s\hat{h}$. Then

$$\begin{aligned} f(y + s\hat{h}) &= \int_{\gamma_0} F \bullet d\vec{x} + \int_{[y, y+s\hat{h}]} F \bullet d\vec{x} \\ &= \text{constant} + \int_0^s F(y + t\hat{h}) \bullet \hat{h} dt. \end{aligned}$$



We immediately obtain from the FTC the formula for the directional derivative

$$\frac{d}{ds}(f(y + s\hat{h})) = F(y + s\hat{h}) \bullet \hat{h}.$$

Set $s = 0$ to achieve

$$Df(y; \hat{h}) = F(y) \bullet \hat{h}.$$

In particular, when $\hat{h} =$ unit coordinate vector \hat{e}_i ,

$$\frac{\partial f}{\partial x_i}(y) = F_i(y).$$

Thus the partial derivatives $\partial f / \partial x_i$ are equal to F_i and are thus continuous; we conclude that f is of class C^1 and $\nabla f = F$.

SECOND PROOF. This time we go straight for the definition of differentiability as found in Section 2E. We have of course a candidate for $\nabla f(y)$, namely $F(y)$. So we are led to consider the quantity

$$f(y+h) - f(y) - F(y) \bullet h$$

for small $h \in \mathbb{R}^n$. Then the definition of f gives

$$\begin{aligned} f(y+h) - f(y) - F(y) \bullet h &= \int_{[y, y+h]} F \bullet d\vec{x} - F(y) \bullet h \\ &= \int_0^1 F(y+th) \bullet h dt - F(y) \bullet h \\ &= \int_0^1 [F(y+th) - F(y)] \bullet h dt. \end{aligned}$$

Given $\epsilon > 0$, the continuity of F at y implies that there exists $\delta > 0$ such that for $\|z - y\| < \delta$ we have $\|F(z) - F(y)\| < \epsilon$. Thus we have for $\|h\| < \delta$ the inequalities

$$\begin{aligned} |f(y+h) - f(y) - F(y) \bullet h| &\leq \int_0^1 |[F(y+th) - F(y)] \bullet h| dt \\ &\stackrel{\text{Schwarz}}{\leq} \int_0^1 \|F(y+th) - F(y)\| \|h\| dt \\ &\leq \int_0^1 \epsilon \|h\| dt \\ &= \epsilon \|h\|. \end{aligned}$$

This is precisely what we needed! We conclude that f is differentiable at y and that $\nabla f(y) = F(y)$. QED

DEFINITION. A vector field which satisfies the above equivalent conditions is said to be a *conservative* vector field. And any function f satisfying 3 is said to be a *potential* function for F .

That theorem is really wonderful: clean statement, easy but significant proof. However, it might be difficult to apply in practice. How would we ever verify the hypothesis 2, for instance? We now turn to a discussion of this aspect of the subject. First, we notice the following elementary

NECESSARY CONDITION. Suppose $F = (F_1, \dots, F_n)$ is a conservative vector field defined on an open set $D \subset \mathbb{R}^n$, and suppose F is of class C^1 . Then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for all } i, j.$$

PROOF. We use the third characterization that $F = \nabla f$ for some potential function f of class C^1 . Since F is assumed to be of class C^1 , we conclude that f is of class C^2 . Therefore, its mixed partial derivatives are equal, and we have immediately

$$\begin{aligned}\frac{\partial F_i}{\partial x_j} &= \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) \\ &= \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) \\ &= \frac{\partial F_j}{\partial x_i}.\end{aligned}$$

QED

DEFINITION. Any vector field satisfying the condition we have just given is said to *have zero curl*. Also it is said to be *irrotational*. These terms will be explained in Chapter 13. In particular, we do not at the present time define the term “curl.”

This necessary condition is certainly simply to apply. For instance, on \mathbb{R}^3 the vector field

$$(y^2z + yz^2, 2xyz + xz^2, xy^2 + xyz)$$

is not conservative, since

$$\begin{aligned}\frac{\partial}{\partial z}(2xyz + xz^2) &= 2xy + 2xz, \\ \frac{\partial}{\partial y}(xy^2 + xyz) &= 2xy + xz\end{aligned}$$

are not equal.

However, the necessary condition is not sufficient, even in case F is of class C^1 . Everyone's favorite counterexample is the following. Use the polar coordinate “function” θ on \mathbb{R}^2 :

$$\begin{aligned}x &= r \cos \theta, \\ y &= r \sin \theta.\end{aligned}$$

Of course, θ is undefined at the origin and is also determined only up to additive integer multiples of 2π . But the *gradient* of θ is completely well defined on $\mathbb{R}^2 - \{0\}$. Using $\theta = \arctan(y/x)$ for example, we have

$$\nabla \theta = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

And that gives us our counterexample. Namely, the vector field

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

is of class C^∞ on $\mathbb{R}^2 - \{0\}$, and

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$

but F is not conservative. For instance F fails to satisfy the criterion 2, as we can detect if we integrate around any circle centered at 0: $x = a \cos t$, $y = a \sin t$ for $0 \leq t \leq 2\pi$ gives

$$\begin{aligned} \int_{\text{circle}} F \bullet d\vec{x} &= \int_0^{2\pi} \left(-\frac{\sin t}{a}, \frac{\cos t}{a} \right) \bullet (-a \sin t, a \cos t) dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt \\ &= 2\pi. \end{aligned}$$

(Of course, we might formally write this computation as

$$\begin{aligned} \int_{\text{circle}} F \bullet d\vec{x} &= \int_{\text{circle}} \nabla \theta \bullet d\vec{x} \\ &= \theta \Big|_{\text{initial point}}^{\text{final point}} \\ &= 2\pi.) \end{aligned}$$

We shall call this nonconservative vector field $\nabla \theta$; the vector field is well defined through the “potential function θ ” is not.

PROBLEM 12–6. Do the differentiations to check that for the vector field $\nabla \theta$ we have

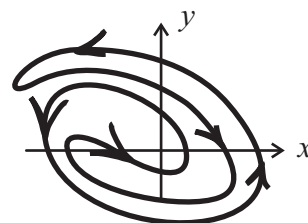
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

And then explain why it is completely unnecessary to do so, that the equation actually holds “automatically.”

PROBLEM 12–7. Let γ be a closed curve in \mathbb{R}^2 which loops around the origin exactly once in the counterclockwise direction:

Show that

$$\int_{\gamma} \frac{-ydx + xdy}{x^2 + y^2} = 2\pi.$$



PROBLEM 12–8. Determine all possible values of the line integral

$$\int_{\gamma} \frac{-ydx + xdy}{x^2 + y^2}$$

for curves γ which start at $(-3, 0)$ and end at $(1, -1)$ and do not pass through the origin.

PROBLEM 12–9. Show that the vector field on $\mathbb{R}^2 - \{0\}$ given by

$$F(x, y) = \left(\frac{y^3}{r^4}, \frac{-xy^2}{r^4} \right),$$

where of course $r^2 = x^2 + y^2$, has zero curl.

PROBLEM 12–10. Show that the vector field of the preceding problem can be expressed in the form

$$F = \frac{1}{2} \nabla \left(\frac{xy}{r^2} - \theta \right).$$

PROBLEM 12–11. Calculate the line integral

$$\int_{\gamma} \frac{y^3 dx - xy^2 dy}{(x^2 + y^2)^2},$$

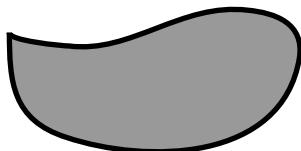
where γ is the counterclockwise ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

F. Sufficiency

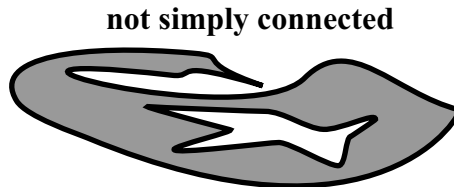
We now want to consider a C^1 vector field on an open subset D of \mathbb{R}^n and investigate whether it is conservative. Of course, we necessarily assume it has zero curl. We then turn to the question of deciding whether it is conservative. We'll give two types of answers.

CASE 1. $n = 2$, and D is simply connected.

The hypothesis indicates that D has no “holes.”

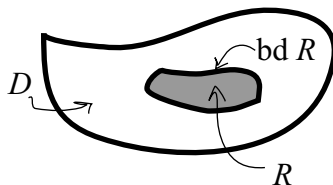


simply connected



not simply connected

Then it follows that zero curl \implies conservative. We can see this from Green's theorem. We verify criterion 2 of Section E. Consider a loop in D , and assume it is of a simple enough nature that it can be realized as the boundary of a nice region R :



Then Green \implies

$$\begin{aligned} \int_{\text{bd}R} F \bullet d\vec{x} &= \int_{\text{bd}R} (F_1 dx + F_2 dy) \\ &= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \\ &= \iint_R 0 dx dy \\ &= 0. \end{aligned}$$

This proves that criterion 2 holds for F , at least for a pretty broad variety of loops. But certainly enough loops to ensure that the proof of $2 \implies 1 \implies 3$ holds.

CASE 2. Any n , and D is all of \mathbb{R}^n or is a rectangle or

We can go directly to determining a potential function by the simple device of integration. Before giving the proof, here is an example of the procedure. Let us take $n = 3$ and

$$F = (2xy - 3yz, x^2 - 3xz, 6z^2 - 3xy).$$

Without checking its curl let us just see whether there may be a potential function $f(x, y, z)$. This would have to satisfy

$$\begin{cases} f_x &= 2xy - 3yz, \\ f_y &= x^2 - 3xz \\ f_z &= 6z^2 - 3xy. \end{cases}$$

Naively integrate the first equation:

$$f = x^2y - 3xyz + g(y, z),$$

where g is a ‘‘constant’’ of integration. Plug this into the second equation to get

$$x^2 - 3xz + g_y = x^2 - 3xz.$$

Thus

$$g_y = 0.$$

Naively integrate this last equation to get

$$g = h(z),$$

where h is the ‘constant’’ of integration. Thus we now have

$$f = x^2y - 3xyz + h(z).$$

Plug this into the third equation to get

$$-3xy + h'(z) = 6z^2 - 3xy.$$

Thus

$$h'(z) = 6z^2.$$

Integrate to get

$$h(z) = 2z^3 + \text{const.}$$

Thus we have succeeded in finding

$$f = x^2y - 3xyz + 2z^3.$$

MORAL. This naive procedure always works in case D is a rectangle or all of \mathbb{R}^n . If F is conservative, it produces a potential function. If F is not conservative, somewhere along the way an impasse will be reached and we'll *learn* that F is not conservative.

Moreover, the general case works in exactly the same way as the example! We can proceed by induction on n .

As an example, let us use the procedure to test whether the vector field

$$F = (2xy - 3yz, x^2 + 3xz, 6z^2 - 3xy)$$

is conservative. We suppose it is, so $F = \nabla f$. Then the equation for $\partial f/\partial x$ gives as before

$$f = x^2y - 3xyz + g(y, z).$$

Then the equation for $\partial f/\partial y$ gives

$$x^2 - 3xz + \frac{\partial g}{\partial y} = x^2 + 3xz,$$

so that

$$\frac{\partial g}{\partial y} = -6xz.$$

This equation is impossible, as $g = g(y, z)$ is independent of x , but the right side $-6xz$ is dependent on x . Thus F is not conservative.

Here is a description of the inductive procedure. We suppose $F = (F_1, \dots, F_n)$ is a C^1 vector field defined on \mathbb{R}^n , and we suppose F satisfies the “zero curl condition.” That is,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for } 1 \leq i, j \leq n.$$

Assume that we have verified the validity of our procedure for dimension $n-1$. We then apply it to the vector field (F_1, \dots, F_{n-1}) on \mathbb{R}^{n-1} , where x_n is just regarded as a parameter. The integration procedure therefore produces a function $f_0(x_1, \dots, x_{n-1}, x_n)$ which satisfies

$$\frac{\partial f_0}{\partial x_i} = F_i \quad \text{for } 1 \leq i \leq n-1.$$

We anticipate that one more integration will give the desired potential function f . So we hope that

$$f(x) = f_0(x) + g(x_n)$$

will do. The required equation is $\partial f / \partial x_n = F_n$, or in other words

$$g'(x_n) = F_n - \frac{\partial f_0}{\partial x_n}.$$

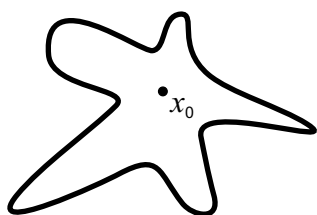
We need to know that the right side of this equation is independent of x_1, \dots, x_{n-1} . So we check its partial derivative with respect to x_i for $i < n$:

$$\begin{aligned} \frac{\partial(F_n - \frac{\partial f_0}{\partial x_n})}{\partial x_i} &= \frac{\partial F_n}{\partial x_i} - \frac{\partial^2 f_0}{\partial x_i \partial x_n} \\ &= \frac{\partial F_n}{\partial x_i} - \frac{\partial^2 f_0}{\partial x_n \partial x_i} \\ &= \frac{\partial F_n}{\partial x_i} - \frac{\partial F_i}{\partial x_n} \end{aligned}$$

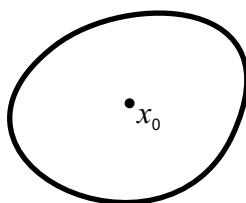
by the inductive assumption. Now the right side is indeed 0, thanks to the curl assumption. So one final integration gives us $g(x_n)$, and, thus, $f(x)$.

G. A slick proof of sufficiency

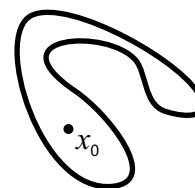
We suppose that F is a C^1 vector field on \mathbb{R}^n , defined on an open set $B \subset \mathbb{R}^n$ which is *star shaped* with respect to one of its points x_0 . This means that for every $x \in B$, the line segment from x_0 to x is contained in B .



star shaped



convex



not star shaped

Any convex open set is star shaped with respect to any of its points.

In what follows we assume (WLOG) that $x_0 = 0$.

If F is conservative with potential f , then from the equation $\nabla f = F$ we can derive a

formula for f . Namely, the FTC gives

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{\partial}{\partial t} f(tx) dt \\ &= f(0) + \int_0^1 (\nabla f)(tx) \bullet x dt \\ &= f(0) + \int_0^1 F(tx) \bullet x dt. \end{aligned}$$

Now turn this around. Suppose F satisfies the zero curl condition and *define*

$$f(x) = \int_0^1 F(tx) \bullet x dt.$$

Notice that f is well defined on B , whether or not F satisfies the extra condition; but the condition is exactly what gives us $\nabla f = F$. We check this by performing a differentiation of f “under the integral sign” as follows:

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\int_0^1 F(tx) \bullet x dt \right) \\ &= \int_0^1 \frac{\partial}{\partial x_i} (F(tx) \bullet x) dt \\ &= \int_0^1 \left[\frac{\partial}{\partial x_i} (F(tx)) \bullet x + F(tx) \bullet \hat{e}_i \right] dt, \end{aligned}$$

the last equality coming from the product rule. Now the chain rule gives

$$\begin{aligned} \frac{\partial}{\partial x_i} (F(tx)) \bullet x &= \left(\frac{\partial F}{\partial x_i} \right) (tx) t \bullet x \\ &= t \sum_{j=1}^n \frac{\partial F_j}{\partial x_i} (tx) x_j \\ &= t \sum_{j=1}^n \frac{\partial F_i}{\partial x_j} (tx) x_j, \end{aligned}$$

the last equality coming from the curl condition. The chain rule now gives

$$\frac{\partial}{\partial x_i} (F(tx)) \bullet x = t \frac{\partial}{\partial t} (F_i(tx)),$$

so the formula for $\frac{\partial f}{\partial x_i}$ becomes

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \int_0^1 \left[t \frac{\partial}{\partial t} (F_i(tx)) + F_i(tx) \right] dt \\ &= \int_0^1 \frac{\partial}{\partial t} [tF_i(tx)] dt \\ &= tF_i(tx) \Big|_{t=0}^{t=1} \\ &= F_i(x).\end{aligned}$$

Thus $\nabla f = F$.