

Chapter 11 Integration on manifolds

We are now almost ready for our concluding chapter on the great theorems of classical vector calculus, the theorems of Green and Gauss and Stokes.

The final thing we need to understand is the correct procedure for integrating over a manifold. Of course, manifolds are typically curved objects, so there are significant issues here that we did not have to face in dealing with integration over (flat) Euclidean space. Even though we have been able to use integration to compute such things as the volume of a ball in \mathbb{R}^n , yet the integration involved is essentially flat. This is reflected in a formula such as

$$d\text{vol}_n = dx_1 dx_2 \dots dx_n.$$

As a specific example, we have for the ball $B(0, r)$ in \mathbb{R}^3 ,

$$\text{vol}_3(B(0, r)) = \frac{4}{3}\pi r^3,$$

but we have not yet even defined what we should mean by the area of the sphere,

$$\text{vol}_2(S(0, r)).$$

It will in fact turn out that there is a very reasonable way to accomplish this task, and it will be based on our knowledge of flat integration.

We choose to employ the *parametric* presentation of manifolds as discussed thoroughly in Section 6E. It is in this context that we shall see how to define volume and integration. Though our definition is to be based on a particular choice of parameters, we shall be able to prove that the integration we define is actually invariant under a change of parameters and is thus intrinsic to the manifold.

We denote by M an m -dimensional manifold contained in \mathbb{R}^n . In what follows we shall first investigate how to define the m -dimensional volume of subsets of M , and after that we shall easily define integrals of real-valued functions defined on M .

We begin with sort of a warm up case, that of one-dimensional manifolds in \mathbb{R}^n .

A. The one-dimensional case

We assume in this section that $m = 1$, so we are dealing essentially with a curve $M \subset \mathbb{R}^n$. We effectively already know exactly how to deal with this case, thanks to Section 2B.

If M is represented parametrically as the image of a one-to-one function of class C^1 ,

$$(a, b) \xrightarrow{F} M,$$

then the length of M is given as

$$\text{vol}_1(M) = \int_a^b \|F'(t)\| dt.$$

Of course, $M \subset \mathbb{R}^n$ and $F'(t) \in \mathbb{R}^n$ and $\|F'(t)\|$ is the norm of the vector $F'(t)$.

We really need to say no more about this definition except to remark that if we imagine the interval (a, b) to be partitioned into small pieces (α, β) , then F sends (α, β) to an approximate interval in \mathbb{R}^n whose length should be approximately $\|F'(\alpha)\|(\beta - \alpha)$. Thus $\|F'(\alpha)\|$ represents a scale factor relating the parameter length $\beta - \alpha$ to a length in \mathbb{R}^n . A useful way to represent this definition is

$$d\text{vol}_1 = \|F'(t)\| dt.$$

As we discussed in Section 2B, the definition of $\text{vol}_1(M)$ is independent of the choice of parametric representation of M .

PROBLEM 11–1. Suppose $M \subset \mathbb{R}^2$ is a curve described in polar coordinates by an equation $r = g(\theta)$, where $a \leq \theta \leq b$. Show that the length of M is

$$\int_a^b \sqrt{g'(\theta)^2 + g(\theta)^2} d\theta.$$

(This formula is usually written as

$$\int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta$$

or even as

$$\int_a^b \sqrt{(dr)^2 + r^2(d\theta)^2}.)$$

PROBLEM 11–2. A certain curve in \mathbb{R}^2 is described in polar coordinates by the equation

$$r = \sqrt{\cos 2\theta}.$$

This “figure eight” curve is called a *lemniscate*. (See Problem 2–6.) Show that its length is equal to

$$\sqrt{\pi} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{2\pi}}.$$

PROBLEM 11–3. Show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \sin^2 \theta}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

(HINT: substitute $x = \sin \theta$ and then $x = t^{1/4}$.)

PROBLEM 11–4. Show that

$$\int_0^{\pi/2} \frac{\sin^2 \theta}{\sqrt{1 + \sin^2 \theta}} d\theta = \sqrt{\pi} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

PROBLEM 11–5. Combine the preceding two problems to show that

$$\int_0^{\pi/2} \sqrt{1 + \sin^2 \theta} d\theta = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \sqrt{\pi} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

PROBLEM 11–6. The two cylinders $x^2 + z^2 = 1$ and $x^2 + y^2 = 1$ intersect in a certain curve. Find its length.

(Answer: 8 times the answer to the preceding problem.)

Finally, we easily extend the definition of the length of M to the more general definition

of integration of a function over M . Suppose then that $M \xrightarrow{f} \mathbb{R}$. Then we write

$$\int_M f d\text{vol}_1 = \int_a^b f(F(t)) \|F'(t)\| dt.$$

In this particular case this is often called an *arc-length integral*, and is written

$$\int_M f ds \quad \text{or} \quad \int_M f d\ell.$$

A frequent application of these integrals is the calculation of the *average* value of f over M , defined to be

$$\bar{f} = \frac{1}{\text{vol}_1(M)} \int_M f d\text{vol}_1.$$

Notice the easy facts

$$\begin{aligned} \overline{f+g} &= \bar{f} + \bar{g}, \\ \overline{cf} &= c\bar{f} \quad \text{if } c \text{ is a constant,} \\ \bar{1} &= 1. \end{aligned}$$

DEFINITION. The *centroid* of the curve M is the point $\bar{x} \in \mathbb{R}^n$ whose coordinates are given by

$$\begin{aligned} \bar{x}_i &= \text{average of } x_i \text{ over } M \\ &= \frac{1}{\text{vol}_1(M)} \int_M x_i d\text{vol}_1. \end{aligned}$$

In other words, \bar{x} is the average position in \mathbb{R}^n of points in M . Note that it is not to be expected that \bar{x} necessarily belongs to M .

PROBLEM 11–7. Calculate the centroid of a semicircle.

PROBLEM 11–8. Calculate the centroid of the “right-half” of the lemniscate of Problem 11–2.

(Answer: $\bar{x}=4\sqrt{\pi}/\Gamma^2(1/4)$, $\bar{y}=0$.)

B. The general case

Amazingly, what we have accomplished in the one-dimensional case generalizes almost immediately to m dimensions. We thus suppose $M \subset \mathbb{R}^n$ is an m -dimensional manifold. We also suppose that M is given by a parametric presentation of class C^1 ,

$$A \xrightarrow{F} \mathbb{R}^n, \quad F(A) = M,$$

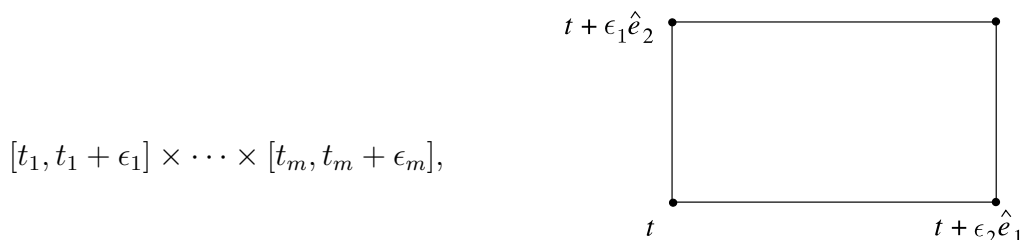
where A is a contented subset of \mathbb{R}^m . For short we may write $x = F(t)$, so that the coordinates of x are given real-valued functions of $t = (t_1, \dots, t_m)$. In our discussion we must have $1 \leq m \leq n$; the case $m = n$ is tantamount to a discussion of integration over \mathbb{R}^n itself.

In every situation our goal is the following: to use the calculus associated with F to find a scale factor $\mathcal{J}(t)$ which converts the m -dimensional “infinitesimal” volume $dt = dt_1 \cdots dt_m$ of the parameter space to the corresponding m -dimensional volume of M . We shall write this correspondence in the form

$$\boxed{d\text{vol}_m = \mathcal{J}(t)dt.}$$

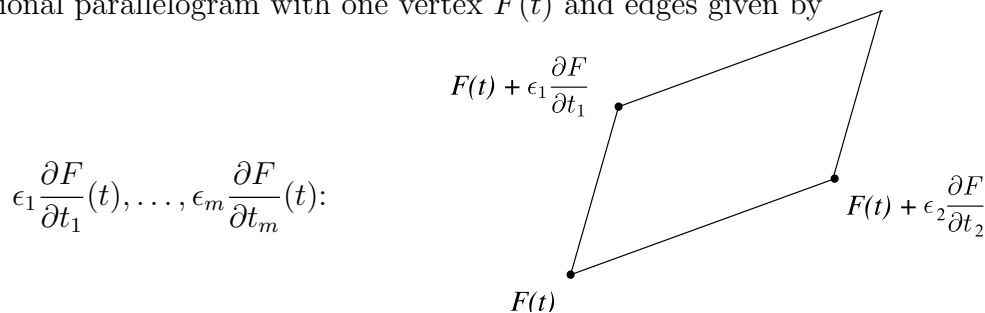
We have chosen the notation $\mathcal{J}(t)$ to remind ourselves that this scale factor is like a Jacobian determinant. In this and other formulas of a similar nature we shall be very insistent upon gaining an understanding of the Jacobian $\mathcal{J}(t)$.

As in the one-dimensional case, what we really need to do is see how F transforms small rectangles in A and in particular what scale factor to multiply by to compute the approximate m -dimensional volume of the image of the rectangle. As usual, let $\hat{e}_1, \dots, \hat{e}_m$ denote the unit coordinate vectors in \mathbb{R}^m . Then consider a special rectangle

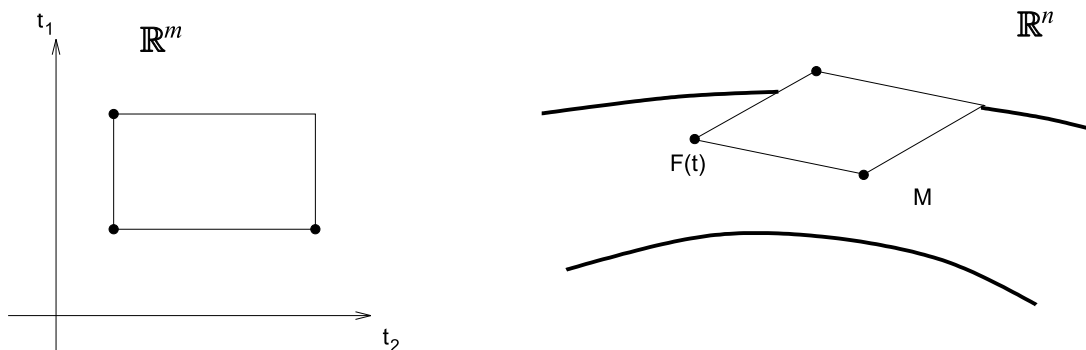


where t is a vertex and the edge lengths are $\epsilon_1, \dots, \epsilon_m$. We think of the ϵ_i 's as small positive numbers.

The corresponding image of this rectangle with respect to F is approximately an m -dimensional parallelogram with one vertex $F(t)$ and edges given by



Of course this flat parallelogram does not lie in the manifold M . However, it is tangent to M at the point $F(t)$.



We know how to compute the m -dimensional volume of this parallelogram: from Section 8C it is

$$\sqrt{\text{Gram}(\epsilon_1 \partial F / \partial t_1, \dots, \epsilon_m \partial F / \partial t_m)} = \epsilon_1 \dots \epsilon_m \sqrt{\text{Gram}(\partial F / \partial t_1, \dots, \partial F / \partial t_m)}.$$

Since the factor $\epsilon_1 \dots \epsilon_m$ is the volume of the rectangle in the parameter space, we see immediately that the scale factor for computing the volume of M is precisely

$$\sqrt{\text{Gram}(\partial F / \partial t_1, \dots, \partial F / \partial t_m)}.$$

Since we think of ϵ_i as dt_i in some sense, and thus $\epsilon_1 \dots \epsilon_m$ is like $dt_1 \dots dt_m = dt$, we are thus led to the

DEFINITION. Given the C^1 parametric presentation of the m -dimensional manifold $M \subset \mathbb{R}^n$

$$A \xrightarrow{F} M,$$

where A is a contented subset of \mathbb{R}^m ,

$$\text{vol}_m(M) = \int_A \sqrt{\text{Gram}(\partial F / \partial t_1, \dots, \partial F / \partial t_m)} dt_1 \dots dt_m.$$

More generally, if f is a real-valued function defined on M , its integral over M is given by

$$\int_M f d\text{vol}_m = \int_A f(F(t)) \sqrt{\text{Gram}(\partial F / \partial t_1, \dots, \partial F / \partial t_m)} dt.$$

We summarize both these equations by simply writing

$$d\text{vol}_m = \sqrt{\text{Gram}(\partial F/\partial t_1, \dots, \partial F/\partial t_m)} dt.$$

This equation gives the scale factor $\mathcal{J}(t)$ explicitly in this general setting.

There is a nice alternate description of the scale factor described above. Namely, use the *Jacobian matrix*

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial t_1} & \cdots & \frac{\partial F_1}{\partial t_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial t_1} & \cdots & \frac{\partial F_n}{\partial t_m} \end{pmatrix}$$

(see Section 2H). This Jacobian matrix is an $n \times m$ matrix, and the matrix product

$$(DF)^t DF$$

is precisely the $m \times m$ Gram matrix $(\partial F/\partial t_i \bullet \partial F/\partial t_j)$. Thus the scale factor in the above definition can also be written in the form

$$\mathcal{J}(t) = \sqrt{\det(DF)^t DF}.$$

EXAMPLES. For the case of curves, $m = 1$, this is the same definition we used before, as $\text{Gram}(\partial F/\partial t) = \text{Gram}(F'(t)) = F'(t) \bullet F'(t)$, so the scale factor is $\|F'(t)\|$.

For the case of 2-dimensional manifolds, we have

$$\text{vol}_2(M) = \int_A \sqrt{\|\partial F/\partial t_1\|^2 \|\partial F/\partial t_2\|^2 - (\partial F/\partial t_1 \bullet \partial F/\partial t_2)^2} dt.$$

Let us examine the sphere $S(0, a)$ of radius a in \mathbb{R}^3 . Let us use spherical coordinates in the usual way, so that

$$F(\varphi, \theta) = (a \sin \varphi \cos \theta, \quad a \sin \varphi \sin \theta, \quad a \cos \varphi).$$

Then

$$\begin{aligned} \partial F/\partial \varphi &= (a \cos \varphi \cos \theta, \quad a \cos \varphi \sin \theta, \quad -a \sin \varphi), \\ \partial F/\partial \theta &= (-a \sin \varphi \sin \theta, \quad a \sin \varphi \cos \theta, \quad 0). \end{aligned}$$

The scale factor is therefore

$$\sqrt{a^2(a \sin \varphi)^2 - 0} = a^2 \sin \varphi.$$

Thus for this sphere we have

$$d\text{vol}_2 = a^2 \sin \varphi d\varphi d\theta.$$

In particular, the area of the sphere is

$$\begin{aligned} \text{vol}_2(S(0, a)) &= \int_0^{2\pi} \int_0^\pi a^2 \sin \varphi d\varphi d\theta \\ &= 4\pi a^2, \end{aligned}$$

a familiar result to be sure.

The example of a right circular cylinder in \mathbb{R}^3 is even easier, but it is so important we should pause to consider it. Let us use the cylinder of radius a with axis given as the z -axis:

$$x^2 + y^2 = a^2.$$

It is convenient to use cylindrical coordinates (no surprise!), which we call θ , t :

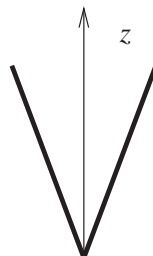
$$\begin{cases} x &= a \cos \theta, \\ y &= a \sin \theta, \\ z &= t. \end{cases}$$

We thus have

$$\begin{aligned} F(\theta, t) &= (a \cos \theta, a \sin \theta, t); \\ \frac{\partial F}{\partial \theta} &= a(-\sin \theta, \cos \theta, 0); \\ \frac{\partial F}{\partial t} &= (0, 0, 1); \\ \mathcal{J}(\theta, t) &= a; \\ d\text{vol}_2 &= a d\theta dt. \end{aligned}$$

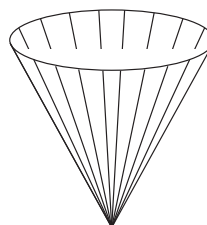
PROBLEM 11–9. Consider the right circular cone as in Problem 6–29:

$$z = \cot \alpha \sqrt{x^2 + y^2}.$$



In terms of cylindrical coordinates θ, z show that for this manifold

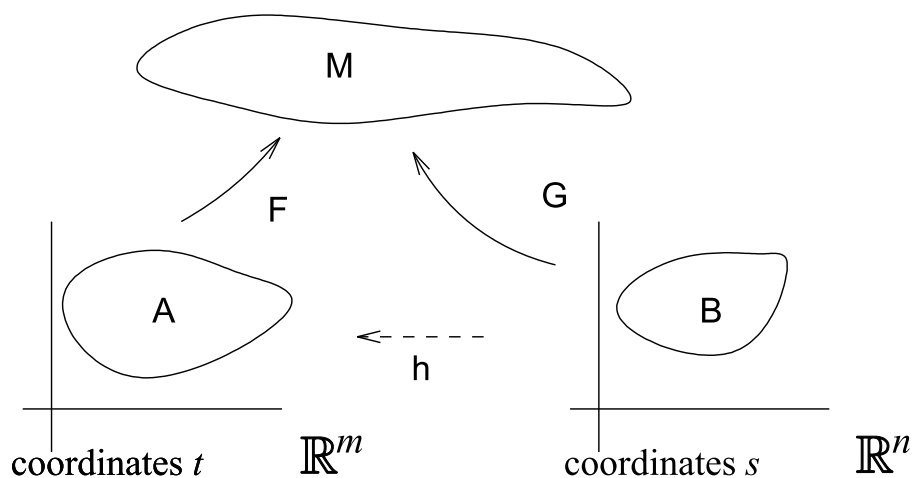
$$d\text{vol}_2 = z \tan \alpha \sec \alpha d\theta dz.$$



Then show that the area of a right circular cone whose base has circumference c and whose slant height is ℓ equals $\frac{1}{2}c\ell$.

Just as we have done in the one-dimensional case of the preceding section, we need to check that our definition is independent of the choice of parametrization of M . In other words, we must verify that if we change parameters we do not change the value of the integral. To be sure, our geometric intuition tells us that all is well, but we should do the work to certify the formulas.

Thus we suppose that M is parametrized in two manners:



There is a parameter change involved here, namely

$$h(s) = F^{-1}(G(s)).$$

That is, $h = F^{-1} \circ G$; we may represent

$$t = h(s),$$

and h is a C^1 function from B onto A and has a C^1 inverse.

THEOREM. *In the above situation the integral*

$$\int_M f d\text{vol}_m$$

has the same value whether computed using F or G .

PROOF. By now we realize that this proof will probably amount to no more than an automatic calculation. And so it does. The scale factor in using G is the square root of

$$\begin{aligned} \det(DG)^t DG &= \det D(F \circ h)^t D(F \circ h) \\ &= \det (((DF) \circ h) Dh)^t ((DF) \circ h) Dh \quad (\text{chain rule}) \\ &= \det(Dh)^t ((DF) \circ h)^t ((DF) \circ h) Dh \\ &= \det(Dh)^t \det[((DF) \circ h)^t ((DF) \circ h)] \det Dh \\ &= (\det(Dh))^2 [\det(DF)^t DF] \circ h. \end{aligned}$$

Thus we obtain

$$\begin{aligned} & \int_B f(G(s)) \sqrt{\det(DG(s))^t DG(s)} ds \\ &= \int_B f(F(h(s))) |\det Dh(s)| \sqrt{\det(DF)^t DF}(h(s)) ds \\ &= \int_A f(F(t)) \sqrt{\det DF(t)^t DF(t)} dt. \end{aligned}$$

This is the desired equation. Notice that the last step used the change of variables formula for integrals over \mathbb{R}^m , $t = h(s)$ and $dt = |\det h'(s)| ds$, which we introduced in Section 10G.

QED

In summary, it should be stressed that the thing which shows that our definition of integration over manifolds makes sense is precisely the change of variables formula of Section 10G.

PROBLEM 11-10. The flat torus in \mathbb{R}^4 is a nice example of a two-dimensional manifold. See Example 3 of Section 5G for the definition. Calculate its two-dimensional volume (its area).

C. Hypermanifolds

We now examine the important special case of a manifold $M \subset \mathbb{R}^n$ of dimension $n-1$. The scale factor for volume calculations has a particularly nice form. We first give a calculation:

LEMMA. $\det(\delta_{ij} + a_i a_j) = 1 + \sum_j a_j^2.$

PROOF. This is a special case of Problem 3-41; nevertheless we provide a proof here. Suppose we are dealing with an $n \times n$ matrix. Then its j^{th} column is $\hat{e}_j + a_j a$, where a is the column vector with coordinates a_1, \dots, a_n . The linearity property of \det (Section 3F), implies that the determinant we want,

$$\det(\hat{e}_1 + a_1 a \quad \hat{e}_2 + a_2 a \quad \cdots \quad \hat{e}_n + a_n a),$$

is equal to the sum of 2^n terms, each term being the determinant of a matrix with either \hat{e}_j or $a_j a$ in the j^{th} column. If a scalar multiple of a appears in at least two columns, the corresponding determinant is zero. Thus only n of the terms of this sum survive, and we

obtain for the above determinant

$$\begin{aligned}
 & \det I + \sum_{j=1}^n \det(\hat{e}_1 \cdots a_j a \cdots \hat{e}_n) \\
 &= 1 + \sum_{j=1}^n a_j \det(\hat{e}_1 \cdots a \cdots \hat{e}_n) \\
 &= 1 + \sum_{j=1}^n a_j \det(\hat{e}_1 \cdots a_j \hat{e}_j \cdots \hat{e}_n) \\
 &= 1 + \sum_{j=1}^n a_j^2.
 \end{aligned}$$

QED

PROBLEM 11–11. For the vectors $\hat{e}_j + a_j \hat{e}_n$, $1 \leq j \leq n-1$, show that

$$\text{Gram}(\hat{e}_1 + a_1 \hat{e}_n, \dots, \hat{e}_{n-1} + a_{n-1} \hat{e}_n) = 1 + \sum_{j=1}^{n-1} a_j^2.$$

Now suppose that the hypermanifold $M \subset \mathbb{R}^n$ is presented as the *graph* of a function $A \xrightarrow{\varphi} \mathbb{R}$, where $A \subset \mathbb{R}^{n-1}$. Then we obtain an explicit parametrization of M with $A \xrightarrow{F} \mathbb{R}^n$, where

$$F(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, \varphi(x_1, \dots, x_{n-1})).$$

Therefore

$$\frac{\partial F}{\partial x_j} = \hat{e}_j + \frac{\partial \varphi}{\partial x_j} \hat{e}_n, \quad 1 \leq j \leq n-1.$$

Problem 11–11 immediately gives

$$\text{Gram}(\partial F / \partial x_1, \dots, \partial F / \partial x_{n-1}) = 1 + \sum_{j=1}^{n-1} (\partial \varphi / \partial x_j)^2.$$

Thus we can compute volume integrals over M in terms of the coordinates x_1, \dots, x_{n-1} , by

$$d\text{vol}_{n-1} = \sqrt{1 + \sum_{j=1}^{n-1} (\partial \varphi / \partial x_j)^2} dx_1 \cdots dx_{n-1}.$$

PROBLEM 11–12. Compute the area of the portion of the surface $z = xy$ contained inside the cylinder $x^2 + y^2 \leq 3$.

(Answer: $14\pi/3$.)

PROBLEM 11–13. Compute the area of the portion of the paraboloid $2z = x^2 + y^2$ contained inside the cylinder $x^2 + y^2 \leq 3$.

We can now immediately generalize this result to hypermanifolds presented implicitly.

THEOREM. Suppose $M \subset \mathbb{R}^n$ is a hypermanifold presented as the level set

$$g(x) = 0,$$

where g is a function on \mathbb{R}^n of class C^1 satisfying $\nabla g \neq 0$ on M . Suppose for instance that $\partial g/\partial x_n \neq 0$. Then integration over M can be calculated by means of the formula

$$dvol_{n-1} = \frac{\|\nabla g\|}{|\partial g/\partial x_n|} dx_1 \cdots dx_{n-1}.$$

PROOF. The implicit function theorem guarantees the local existence of a function $\mathbb{R}^{n-1} \xrightarrow{\varphi} \mathbb{R}$ such that

$$g(x) = 0 \iff x_n = \varphi(x_1, \dots, x_{n-1}).$$

Furthermore, differentiating the equation

$$g(x_1, \dots, x_{n-1}, \varphi(x_1, \dots, x_{n-1})) = 0$$

gives the formula

$$\partial \varphi / \partial x_j = -\frac{\partial g / \partial x_j}{\partial g / \partial x_n}.$$

Thus the scale factor found above in terms of φ can be manipulated to give

$$\begin{aligned} \sqrt{1 + \sum_{j=1}^{n-1} (\partial \varphi / \partial x_j)^2} &= \sqrt{1 + \sum_{j=1}^{n-1} (\partial g / \partial x_j)^2 / (\partial g / \partial x_n)^2} \\ &= \frac{1}{|\partial g / \partial x_n|} \sqrt{(\partial g / \partial x_n)^2 + \sum_{j=1}^{n-1} (\partial g / \partial x_j)^2}. \end{aligned}$$

QED

EXAMPLE. Let the manifold be the sphere $S(0, a) \subset \mathbb{R}^n$. This is an $(n - 1)$ -dimensional manifold and is described implicitly by the equation

$$\|x\|^2 - a^2 = 0.$$

For the function we have

$$\nabla g = 2x, \quad \partial g / \partial x_n = 2x_n.$$

In order to cope with $\partial g / \partial x_n \neq 0$ we shall find the volume of the upper hemisphere $x_n > 0$ and then double the answer. Thus

$$\begin{aligned} \text{vol}_{n-1}(S(0, a)) &= 2 \cdot \int \frac{\|2x\|}{2x_n} dx_1 \dots dx_{n-1} \\ &= 2 \int \frac{a}{x_n} dx_1 \dots dx_{n-1} \\ &= 2 \int \frac{a}{\sqrt{a^2 - x_1^2 - \dots - x_{n-1}^2}} dx_1 \dots dx_{n-1}, \end{aligned}$$

where the integration region is the ball $x_1^2 + \dots + x_{n-1}^2 < a^2$ in \mathbb{R}^{n-1} . From Problem 10–49 we therefore obtain

$$\text{vol}_{n-1}(S(0, a)) = 2(n-1)\alpha_{n-1} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r^{n-2} dr.$$

Substitute $r = a\sqrt{t}$ to obtain

$$\begin{aligned} \text{vol}_{n-1}(S(0, a)) &= (n-1)\alpha_{n-1} \int_0^1 \frac{1}{\sqrt{1-t}} a^{n-2} t^{\frac{n-2}{2}} a t^{-\frac{1}{2}} dt \\ &= (n-1)\alpha_{n-1} a^{n-1} \int_0^1 \frac{1}{\sqrt{1-t}} t^{\frac{n-3}{2}} dt \\ &= (n-1)\alpha_{n-1} a^{n-1} B\left(\frac{n-1}{2}, \frac{1}{2}\right) \\ &= (n-1) \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} a^{n-1} \frac{\Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} a^{n-1}. \end{aligned}$$

We record this result for later reference:

$$\text{vol}_{n-1}(S(0, 1)) = \frac{2\pi^{n/2}}{\Gamma(n/2)} = n\alpha_n.$$

For instance,

$$\begin{aligned}\text{vol}_2(S(0, 1)) &= 4\pi, \\ \text{vol}_3(S(0, 1)) &= 2\pi^2, \\ \text{vol}_4(S(0, 1)) &= \frac{8\pi^2}{3}.\end{aligned}$$

EXAMPLE. Here we simply reconsider the sphere $S(0, a)$ in \mathbb{R}^3 . In Section B we calculated the area formula

$$d\text{vol}_2 = a^2 \sin \varphi d\varphi d\theta$$

in terms of the usual spherical coordinates. It is quite interesting to think of this result also in terms of *cylindrical* coordinates for the sphere: $x^2 + y^2 = a^2 - z^2$, so that

$$\begin{cases} x &= \sqrt{a^2 - z^2} \cos \theta, \\ y &= \sqrt{a^2 - z^2} \sin \theta, \\ z &= z. \end{cases}$$

Of course we assume here that $-a < z < a$. Then we can calculate $\mathcal{J}(z, \theta)$ from scratch, or we can instead observe that in terms of the angle φ we have $z = a \cos \varphi$; thus $dz = a \sin \varphi d\varphi$ (we've included the absolute value). The result is that

$$d\text{vol}_2 = adz d\theta.$$

This is a truly fascinating result. The expression for this area is exactly that of the circumscribed cylinder

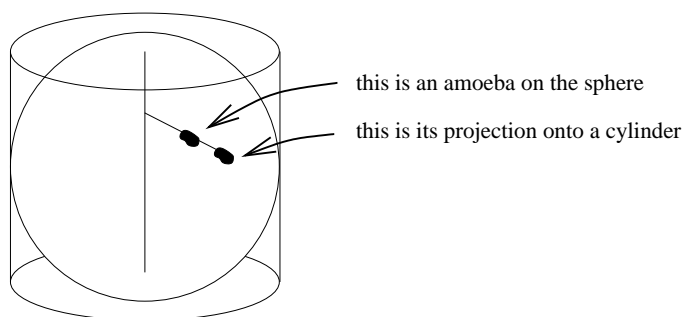
$$x^2 + y^2 = a^2!$$

An interesting application of this result has to do with constructing maps of the sphere onto planes. If we project the sphere onto the cylinder by moving in horizontal straight lines from the z -axis, then *area is preserved!* The precise formula for this map is

$$F(x, y, z) = \left(\frac{ax}{\sqrt{a^2 - z^2}}, \frac{ay}{\sqrt{a^2 - z^2}}, z \right).$$

Its inverse is

$$F^{-1}(x, y, z) = \left(\sqrt{a^2 - z^2} \frac{x}{a}, \sqrt{a^2 - z^2} \frac{y}{a}, z \right).$$

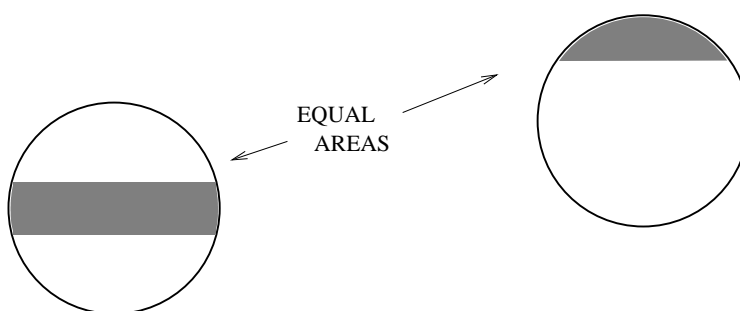


Then the map onto a plane is obtained by cutting the cylinder along a generator and unrolling it.

An interesting and curious consequence comes from the above observation. Namely, a *zone* on the sphere is defined as a set of the form

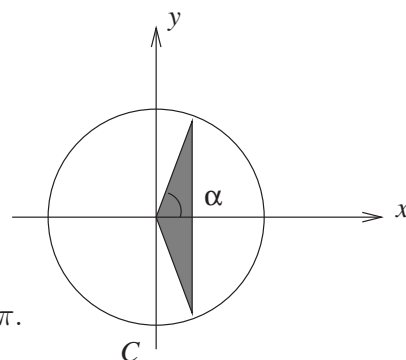
$$\{(x, y, z) \mid x^2 + y^2 + z^2 = a^2, \quad z_1 \leq z \leq z_2\}.$$

Here $-a \leq z_1 < z_2 \leq a$. The area of this zone is precisely $2\pi a(z_2 - z_1)$, as that is the corresponding area of the circumscribed cylinder. Thus the area of a zone depends only on its “height” $z_2 - z_1$, and not on its location.



PROBLEM 11–14. Let H be the unit hemisphere in \mathbb{R}^3 described as $x^2 + y^2 + z^2 = 1$, $0 \leq z \leq 1$. Let C be the boundary equator $x^2 + y^2 = 1$, $z = 0$. Let $0 \leq \alpha \leq \frac{\pi}{2}$ and let M be the portion of H which lies over the triangle shown in the figure. That is, in standard spherical coordinates M is described by

$$\begin{cases} 0 \leq \varphi \leq \frac{\pi}{2}, \\ -\alpha \leq \theta \leq \alpha, \\ \sin \varphi \cos \theta \leq \cos \alpha. \end{cases}$$



Show that the area of M equals

$$\text{vol}_2(M) = 2\alpha + \pi \cos \alpha - \pi.$$

(HINT: do no integration.)

PROBLEM 11–15. (This problem is based on Problem B3 from the William Lowell Putnam Mathematical Competition for 1998.) Continue with the hemisphere H of the preceding problem. Let P be a regular n -gon inscribed in the equator C . Determine the surface area of that portion of H lying over the planar region inside P .

(Answer: $2\pi - n\pi + n\pi \cos \pi/n$)

PROBLEM 11–16. Find the area of the portion of the surface of the cylinder $x^2 + y^2 = 1$ in \mathbb{R}^3 which is contained inside the cylinder $x^2 + z^2 \leq 1$.

(Answer: 8)

PROBLEM 11–17. Find the area of the portion of the surface of the cylinder $x^2 + y^2 = 1$ in \mathbb{R}^3 which is contained in the intersection of the two cylinders $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$.

(Answer: $16 - 8\sqrt{2}$)

PROBLEM 11–18*. A surface $M \subset \mathbb{R}^3$ is described implicitly by the equation

$$(x^2 + y^2 + z^2)^2 = x^2 - y^2.$$

Prove that this is actually a two-dimensional manifold at each point except the origin. Compute its area.

(Answer: $\pi^2/2$)

PROBLEM 11–19. Consider these two surfaces in \mathbb{R}^3 : the unit sphere $x^2 + y^2 + z^2 = 1$ and the right circular cylinder $x^2 + y^2 = y$.

- Find the area of the portion of the unit sphere that is inside the cylinder.
- Find the area of the portion of the cylinder that is inside the sphere.

PROBLEM 11–20. Let M be the “elliptical cone” in \mathbb{R}^3 defined by the equation

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

and the inequality $0 \leq z \leq h$. Show that the centroid of M is $(0, 0, 2h/3)$. (You will *not* be able to calculate the area of M .)

D. The Cauchy-Binet determinant theorem

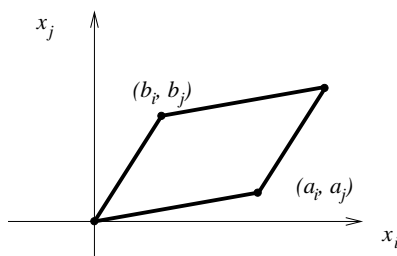
This section is not very important for our development of integration, but is so fascinating that it should be included.

We start with an interpretation of the formula from Section 7B: for any two vectors $a, b \in \mathbb{R}^n$

$$\text{Gram}(a, b) = \sum_{i < j} (a_i b_j - a_j b_i)^2.$$

We know of course that the left side equals the square of the area of the parallelogram P with vertices $0, a, b,$ and $a + b$ (Section 8C). Now consider the orthogonal projection of P onto the coordinate space \mathbb{R}^2 , namely the $x_i - x_j$ plane. This projection $P(i, j)$ is the parallelogram with vertices

$$(0, 0), \quad (a_i, a_j), \quad (b_i, b_j), \quad \text{and} \quad (a_i + b_i, a_j + b_j).$$



The square of its area is of course the corresponding Gram determinant,

$$\det \begin{pmatrix} a_i^2 + a_j^2 & a_i b_i + a_j b_j \\ a_i b_i + a_j b_j & b_i^2 + b_j^2 \end{pmatrix} = (a_i b_j - a_j b_i)^2.$$

Thus we have the interesting fact that the square of the area of P is equal to the sum of the squares of the areas of all the projections $P(i, j)$, for $1 \leq i < j \leq n$. This is sort of a Pythagorean theorem for two-dimensional parallelograms in \mathbb{R}^n .

The above result is a special case of the theorem of this section. Before discussing it let us see what our special case implies about the change of variables formula for two-dimensional manifolds. We use the parametric presentation of $M \subset \mathbb{R}^n$, but instead of naming the presentation function we simply denote its components as x_1, \dots, x_n regarded as functions of t_1, t_2 . Then we have from the above equation

$$(\mathcal{J}(t_1, t_2))^2 = \sum_{i < j} \left(\det \begin{pmatrix} \partial x_i / \partial t_1 & \partial x_i / \partial t_2 \\ \partial x_j / \partial t_1 & \partial x_j / \partial t_2 \end{pmatrix} \right)^2.$$

In terms of the notation mentioned in Section 10I, this says

$$(\mathcal{J}(t_1, t_2))^2 = \sum_{i < j} \left(\frac{\partial(x_i, x_j)}{\partial(t_1, t_2)} \right)^2.$$

This makes the volume formula look like a rather natural generalization of the formula for changing variables in an ordinary integral over \mathbb{R}^2 :

$$d\text{vol}_2 = \sum_{i < j} \left\{ \left(\frac{\partial(x_i, x_j)}{\partial(t_1, t_2)} \right)^2 \right\}^{1/2} dt_1 dt_2.$$

The above result does not appear to be particularly useful for calculations, but its elegance is very interesting. The algebraic generalization we now present is likewise not something we shall be using, but we present it here to display a beautiful piece of algebra.

Instead of $m = 2$, we now consider the general case of $1 \leq m \leq n$. For the present discussion we shall use the notation u to stand for a generic m -tuple of increasing integers in the range $[1, n]$:

$$u = (u_1, \dots, u_m), \quad 1 \leq u_1 < u_2 < \dots < u_m \leq n.$$

Given an $m \times n$ matrix A and an index u , we can construct a (square) $m \times m$ matrix $A(u)$ by using only the *columns* of A corresponding to u :

$$A(u) = \begin{pmatrix} a_{1u_1} & \dots & a_{1u_m} \\ \vdots & & \vdots \\ a_{mu_1} & \dots & a_{1u_m} \end{pmatrix}.$$

Likewise, given an $n \times m$ matrix B we construct $B(u)$ by selecting only the *rows* of B corresponding to u :

$$B(u) = \begin{pmatrix} b_{u_1 1} & \dots & b_{u_1 m} \\ \vdots & & \vdots \\ b_{u_m 1} & \dots & b_{u_m m} \end{pmatrix}.$$

Here is the amazing result:

THEOREM. *If $1 \leq m \leq n$ and A is an $m \times n$ matrix and B is an $n \times m$ matrix, then*

$$\det AB = \sum_u \det A(u) \det B(u).$$

PROOF. The ij entry of AB is of course

$$\sum_k a_{ik} b_{kj},$$

where the sum extends for $1 \leq k \leq n$. Therefore we can write the j^{th} column of AB in the form

$$(AB)_{\text{column } j} = \sum_k b_{kj}(A)_{\text{column } k}.$$

Now we use the linearity of \det with respect to its columns, the multilinear property of Section 3F. For each column of AB we need a distinct summation index; so in the above formula for the j^{th} column we change k to k_j . Then we have

$$\det AB = \sum_{k_1, \dots, k_m} b_{k_1 1} \cdots b_{k_m m} \det ((A)_{\text{col } k_1} \cdots (A)_{\text{col } k_m}).$$

In this huge sum each k_j runs from 1 to n .

Now think about the determinants in the sum. If two of the k_j 's are equal, the corresponding matrix has two identical columns so the determinant is zero. Thus we may restrict the sum to *distinct* k_1, \dots, k_m . These distinct choices, if reordered, correspond to the various increasing sets of indices u . For a given u , there are $m!$ distinct orderings k_1, \dots, k_m , and each determinant in the sum will be $\pm \det A(u)$. Thus we can rewrite the above formula as

$$\det AB = \sum_u f(B, u) \det A(u),$$

where $f(B, u)$ just stands for some number that depends only on B and u (and *not* on A).

Now it's easy to compute each coefficient by just picking a particular A . Given a particular u choose A to be that matrix defined by

$$\begin{cases} a_{iu_j} &= \delta_{ij}, \\ a_{ik} &= 0 \quad \text{if } k \neq u_1, \dots, u_m. \end{cases}$$

That is, the u_j^{th} column of A is the coordinate vector $e_j \in \mathbb{R}^n$, and all other columns of A are 0. Thus all $\det A(u')$ except $u' = u$ are zero, and

$$\begin{aligned} \det AB &= f(B, u) \det A(u) \\ &= f(B, u) \det I \\ &= f(B, u). \end{aligned}$$

On the other hand the ij entry of AB is

$$\begin{aligned} \sum_k a_{ik} b_{kj} &= a_{iu_1} b_{u_1 j} + \cdots + a_{iu_m} b_{u_m j} \\ &= \delta_{i1} b_{u_1 j} + \cdots + \delta_{im} b_{u_m j} \\ &= b_{u_i j}. \end{aligned}$$

That is, $AB = B(u)$. This proves that $f(B, u) = \det B(u)$.

QED

REMARKS. 1. This result in case $m = n$ just asserts that for square matrices $\det AB = \det A \det B$. Thus we have given a third proof of this crucial property of determinants!

2. This proof should remind you of the proof of the lemma of Section C. In fact, in that notation

$$(\delta_{ij} + a_i a_j) = B^t B,$$

where

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ a_1 & a_2 & & a_n \end{pmatrix}$$

is an $(n+1) \times n$ matrix. Note that

$$B((1, \dots, n)) = I,$$

$$B((1, \dots, k-1, k+1, \dots, n+1)) = \left. \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix} \right\}^{k^{\text{th}} \text{ row omitted}},$$

so

$$\det B((1, \dots, n)) = 1,$$

$$\det B((1, \dots, k-1, k+1, \dots, n+1)) = \pm a_k.$$

Thus the lemma is a special case of the theorem.

3. In general, in case $A = B^t$, we have for $m \times n$ matrices with $1 \leq m \leq n$,

$$\det AA^t = \sum_u (\det A(u))^2.$$

4. Consider vectors x, y, u, v in \mathbb{R}^3 and define

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

$$B = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} x \bullet u & x \bullet v \\ y \bullet u & y \bullet v \end{pmatrix},$$

so the theorem gives

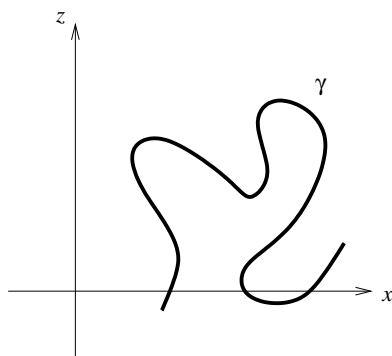
$$\begin{aligned} \det \begin{pmatrix} x \bullet u & x \bullet v \\ y \bullet u & y \bullet v \end{pmatrix} &= \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \\ &+ \det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} \det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix} + \det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}. \end{aligned}$$

This is precisely the result of Problem 7–3, the generalization of Lagrange's identity.

E. Miscellaneous applications

1. Surfaces of revolution

We are going to examine a special class of surfaces in \mathbb{R}^3 obtained by revolving about an axis. We may as well use the z -axis as the axis of revolution. We then start with a curve γ of class C^1 lying in the $x - z$ plane, where all points of γ have positive x -coordinates.



We revolve the points of γ about the z -axis through a full angle of 360° and thus obtain a 2-dimensional manifold (a surface) M . We want to investigate integration on M .

Let $\gamma = \gamma(t)$ be a parametrization, where $a \leq t \leq b$. We assume γ does not intersect itself, except that perhaps γ is a *closed* curve ($\gamma(a) = \gamma(b)$). We shall designate the coordinates of γ as

$$\gamma(t) = (x(t), z(t)).$$

Thus, $x(t) > 0$.

Let θ denote the angle of rotation about the z -axis. Then M can be parametrized as follows:

$$F(t, \theta) = (x(t) \cos \theta, x(t) \sin \theta, z(t)).$$

This is a representation using cylindrical coordinates, where the distance from the axis is $x(t)$. Then we compute easily

$$\begin{aligned} \frac{\partial F}{\partial t} &= (x'(t) \cos \theta, x'(t) \sin \theta, z'(t)), \\ \frac{\partial F}{\partial \theta} &= x(t)(-\sin \theta, \cos \theta, 0). \end{aligned}$$

These tangent vectors are orthogonal, so the corresponding Gram determinant is

$$\left\| \frac{\partial F}{\partial t} \right\|^2 \left\| \frac{\partial F}{\partial \theta} \right\|^2 = (x'(t)^2 + z'(t)^2)x(t)^2.$$

Thus the Jacobian factor for $d\text{vol}_2$ is

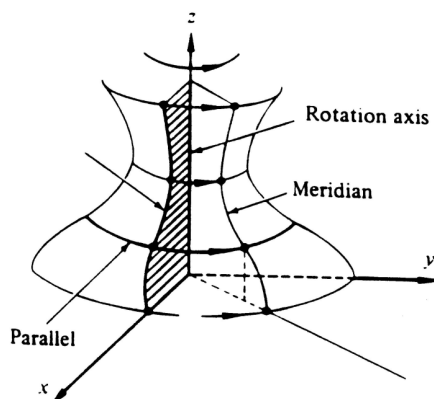
$$\mathcal{J}(t, \theta) = x(t)\sqrt{x'(t)^2 + z'(t)^2}.$$

Hence we obtain for instance

$$\begin{aligned} \text{area}(M) &= \text{vol}_2(M) \\ &= \int_a^b \int_0^{2\pi} x(t)\sqrt{x'(t)^2 + z'(t)^2} d\theta dt \\ &= 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2} dt. \end{aligned}$$

Another way of saying this is

$$\text{area}(M) = 2\pi \int_{\gamma} x ds.$$



A surface of revolution.

PROBLEM 11–21. Instead of revolving a curve, we can revolve a two-dimensional region around an axis. Suppose that R is such a contented subset of the $x - z$ plane and again suppose that for all $(x, z) \in R$, $x > 0$. Then revolve R around the z -axis through 360° , obtaining the *solid* of revolution D :

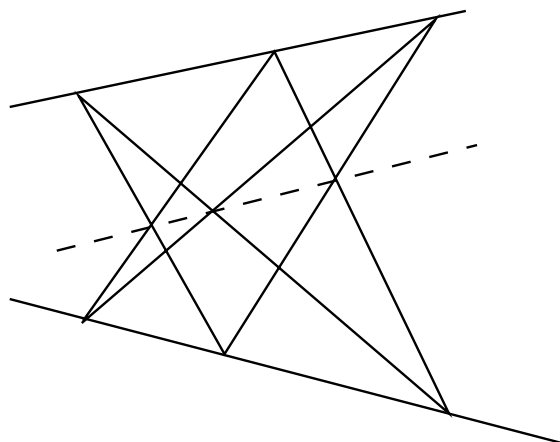
$$D = \{(x \cos \theta, x \sin \theta, z) \mid (x, z) \in R, 0 \leq \theta \leq 2\pi\}.$$

Prove that

$$\text{vol}_3(D) = 2\pi \int_R x dx dz.$$

2. Pappus' theorems

Pappus of Alexandria lived about 290–350, and has been described as the last of the great Greek geometers. Here is one of his theorems, which is known as the basis of modern projective geometry:



Pappus's theorem states:
 If the vertices of a hexagon lie alternately on two lines, then the meets of opposite sides are collinear.

Just for fun, you might try proving this for yourself.

The theorems of Pappus we want to discuss are about areas and volumes of figures of revolution in \mathbb{R}^3 . (We shall of course prove them with calculus).

First consider the case of a surface of revolution, using the notation introduced above. Then

$$\text{area}(M) = 2\pi \int_{\gamma} x ds.$$

But also the *centroid* of γ has its x -coordinate defined by

$$\begin{aligned} \bar{x} &= \frac{\int_{\gamma} x ds}{\int_{\gamma} ds} \\ &= \frac{\int_{\gamma} x ds}{L}, \end{aligned}$$

where $L =$ the length of the curve γ . Thus we obtain Pappus' theorem,

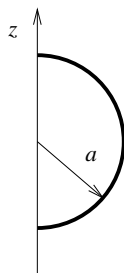
$$\text{area}(M) = 2\pi\bar{x}L.$$

Notice that $2\pi\bar{x}$ is the distance the centroid of γ travels in one rotation. Thus Pappus' theorem can be stated in words as

The area of a surface of revolution equals the product of the length of the initial curve and the distance traveled around the axis by its centroid.

A great example is provided by a torus of revolution. Using the notation of Section 5G, where the center of the initial circle of radius a was located at distance $b > a$ from the axis, the resulting area is given by $2\pi a \cdot 2\pi b = 4\pi^2 ab$.

Sometimes the Pappus theorem can be used “backwards.” For example, suppose we revolve a semicircle around the z -axis as shown:



The resulting surface is a sphere of radius a . Since we already know the area of the sphere, we obtain from Pappus,

$$4\pi a^2 = \pi a \cdot 2\pi \bar{x},$$

and we have found the centroid of a semicircle:

$$\bar{x} = \frac{2a}{\pi}.$$

PROBLEM 11–22. Prove another Pappus theorem for a solid of revolution, and express the result in words as we did above.

PROBLEM 11–23. Use Pappus to find the centroid of a semiball in \mathbb{R}^3 .

3. Tubes

There is an interesting extension of Pappus’ theorems to figures in \mathbb{R}^3 we might describe as “tubes.” Start with a nonselfintersecting C^1 curve γ in \mathbb{R}^3 , parametrized as $\gamma(t)$, $a \leq t \leq b$. Then imagine constructing a circle of radius r centered at each point $\gamma(t)$ of the curve, and orthogonal to the curve. To make sense of the orthogonality, we of course require the tangent vector $\gamma'(t) \neq 0$. Then it can be shown that if r is sufficiently small, “self-intersections” do not occur, and the resulting set is an actual two-dimensional manifold. We want to find its area. In fact, it equals the length of γ times $2\pi r$.

In order to prove this with calculus, we need to construct an orthogonal frame at each point $\gamma(t)$. For the first vector of the frame we can use $\gamma'(t)$. The other two vectors may be chosen rather arbitrarily. Let us call such a choice $u(t)$, $v(t)$, where $u(t)$ and $v(t)$ are unit vectors in \mathbb{R}^3 , and the three vectors,

$$\gamma'(t), \quad u(t), \quad v(t),$$

are mutually orthogonal. We also need to know that $u(t)$ and $v(t)$ are of class C^1 . (One way to insure this is to assume γ is of class C^2 and take $u(t)$ as the unit vector in the same direction as

$$\frac{d}{dt} \frac{\gamma'(t)}{\|\gamma'(t)\|},$$

and then

$$v(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \times u(t).$$

Now notice that differentiating the equation $u \bullet u = 1$ produces $u' \bullet u = 0$. Thus we may express u' as a linear combination of γ' , u , and v , and the result looks like this:

$$u'(t) = a(t)\gamma'(t) + c(t)v(t).$$

Likewise,

$$v'(t) = b(t)\gamma'(t) + d(t)u(t).$$

Notice also that $u \bullet v = 0$ implies $u' \bullet v + u \bullet v' = 0$, so that $c(t) + d(t) = 0$. Thus we have the formulas

$$\begin{cases} u' &= a\gamma' + cv, \\ v' &= b\gamma' - cu. \end{cases}$$

Now we present a parametrization of our tube:

$$F(t, \theta) = \gamma(t) + r \cos \theta u(t) + r \sin \theta v(t).$$

We calculate as follows:

$$\begin{aligned} \frac{\partial F}{\partial t} &= \gamma' + r \cos \theta (a\gamma' + cv) + r \sin \theta (b\gamma' - cu) \\ &= (1 + ar \cos \theta + br \sin \theta)\gamma' - cr \sin \theta u + cr \cos \theta v, \\ \frac{\partial F}{\partial \theta} &= -r \sin \theta u + r \cos \theta v. \end{aligned}$$

Thus we have

$$\left\| \frac{\partial F}{\partial \theta} \right\|^2 = r^2$$

and

$$\frac{\partial F}{\partial t} \bullet \frac{\partial F}{\partial \theta} = cr^2 \sin^2 \theta + cr^2 \cos^2 \theta = cr^2.$$

Thus the Gram determinant is therefore

$$\begin{aligned} \mathcal{J}(t, \theta)^2 &= \left\| \frac{\partial F}{\partial t} \right\|^2 r^2 - (cr^2)^2 \\ &= \{(1 + ar \cos \theta + br \sin \theta)^2 \|\gamma'\|^2 + c^2 r^2\} r^2 - c^2 r^4 \\ &= (1 + ar \cos \theta + br \sin \theta)^2 \|\gamma'\|^2 r^2. \end{aligned}$$

How nice! If r is sufficiently small that $1 + ar \cos \theta + br \sin \theta$ is always positive, then

$$\mathcal{J}(t, \theta) = (1 + ar \cos \theta + br \sin \theta) \|\gamma'\| r.$$

We conclude that

$$\begin{aligned} \text{area}(M) &= r \int_a^b \int_0^{2\pi} (1 + a(t)r \cos \theta + b(t)r \sin \theta) \|\gamma'(t)\| dt \\ &= r \int_a^b 2\pi \|\gamma'(t)\| dt \\ &= 2\pi r \cdot \text{length of } \gamma. \end{aligned}$$

PROBLEM 11–24. Perform the analogous computation for the 3-dimensional volume of a “solid” tube in \mathbb{R}^3 , using the parametrization given above written now as

$$F(t, \theta, r) = \gamma(t) + r \cos \theta u(t) + r \sin \theta v(t),$$

where the new variable r lies in the interval $0 < r < r_0$.

4. Ellipsoids

You probably know that the arc length of an ellipse cannot be expressed in elementary terms, unless of course the ellipse is merely a circle. In fact, it is for this very reason that a class of special functions has been defined, the *elliptic integrals*. Here is one example:

DEFINITION. The *complete elliptic integral of the second kind* is the function E given by

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} dt.$$

For our purposes, we restrict attention to $0 \leq k \leq 1$. Of course we have the elementary values

$$E(0) = \frac{\pi}{2}, \quad E(1) = 1.$$

PROBLEM 11–25. Find one of the few known “elementary” values of E :

$$E\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)} + \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)$$

(see Problem 11–5).

PROBLEM 11–26. Prove that the length of an ellipse is equal to

$$4 \cdot (\text{semimajor axis}) \cdot E(\text{eccentricity}).$$

We should expect that calculating the area of an ellipsoid in \mathbb{R}^3 would lead to even more complicated integrals. Indeed it does. However, there is quite a surprise: the case of ellipsoids of revolution is completely elementary. The results are given in

PROBLEM 11–27. Consider the ellipsoid of revolution in \mathbb{R}^3 ,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Oblate spheroid: Show that if $a > b$, then the area is equal to

$$2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}} \log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right).$$

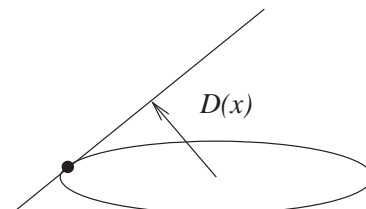
Prolate spheroid: Find the area in case $a < b$ (notice that the above formula doesn't make sense in this case).

Although the length of an ellipse in \mathbb{R}^2 and the area of an ellipsoid in \mathbb{R}^3 are not elementary integrals, there happen to be associated integrals that are. Here is an example:

PROBLEM 11–28. Let $M \subset \mathbb{R}^n$ be the ellipsoid described implicitly as

$$\frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} = 1.$$

Through any $x \in M$ construct the hyperplane which is tangent to M at x . Let $D(x)$ = the distance from this hyperplane to the origin.



a. Prove that

$$D(x) = \frac{1}{\sqrt{\frac{x_1^2}{a_1^4} + \cdots + \frac{x_n^2}{a_n^4}}}$$

b. Calculate explicitly

$$\int_M D d\text{vol}_{n-1}.$$