Chapter 1 Euclidean space

A. The basic vector space

We shall denote by $\mathbb{R}$ the field of real numbers. Then we shall use the Cartesian product $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ of ordered n-tuples of real numbers (n factors). Typical notation for $x \in \mathbb{R}^n$ will be

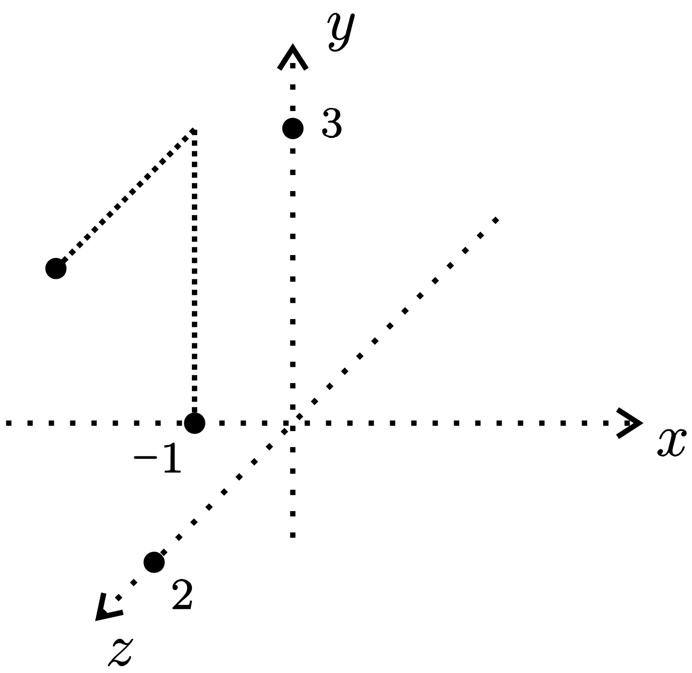
$$x = (x_1, x_2, \ldots, x_n).$$

Here x is called a *point* or a *vector*, and x_1, $x_2, \ldots, x_n$ are called the *coordinates* of x. The natural number n is called the *dimension* of the space. Often when speaking about $\mathbb{R}^n$ and its vectors, real numbers are called *scalars*.

Special notations:

$\mathbb{R}^1$	x
$\mathbb{R}^2$	$x = (x_1, x_2)$ or $p = (x, y)$
$\mathbb{R}^3$	$x = (x_1, x_2, x_3)$ or $p = (x, y, z)$.

We like to draw pictures when $n = 1$, 2, 3; e.g. the point $(-1, 3, 2)$ might be depicted as



We *define* algebraic operations as follows: for x, $y \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n); \\ ax &= (ax_1, ax_2, \dots, ax_n); \\ -x &= (-1)x = (-x_1, -x_2, \dots, -x_n); \\ x - y &= x + (-y) = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n). \end{aligned}$$

We also define the *origin* (a/k/a the point *zero*)

$$0 = (0, 0, \dots, 0).$$

(Notice that 0 on the left side is a vector, though we use the same notation as for the scalar 0.)

Then we have the easy facts:

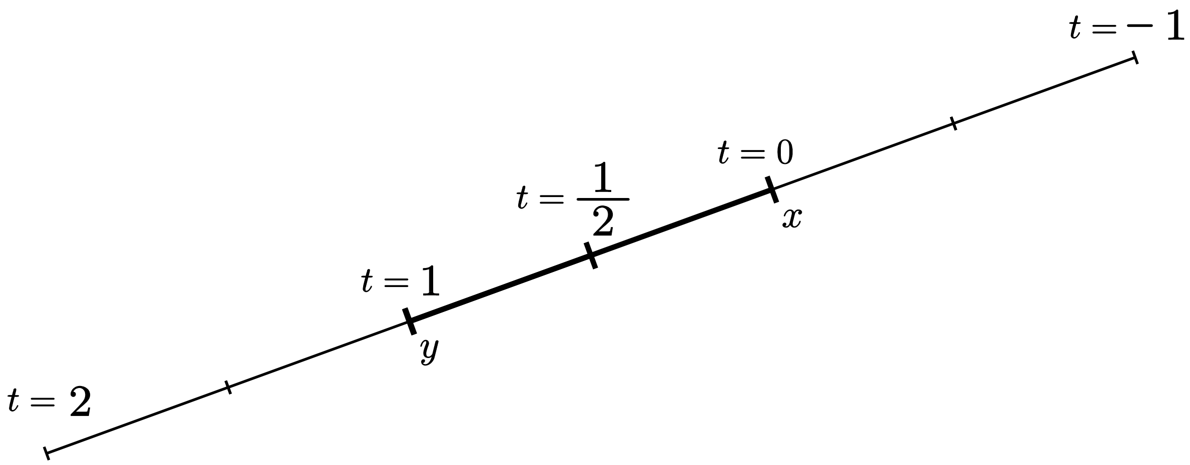
$$\begin{aligned} x + y &= y + x; \\ (x + y) + z &= x + (y + z); \\ 0 + x &= x; \\ x - x &= 0; \\ 1x &= x; \\ (ab)x &= a(bx); \\ a(x + y) &= ax + ay; \\ (a + b)x &= ax + bx; \\ 0x &= 0; \\ a0 &= 0. \end{aligned}$$

in other words all the "usual" algebraic rules are valid if they make sense

Schematic pictures can be very helpful. One nice example is concerned with the *line* determined by x and y (distinct points in $\mathbb{R}^n$). This line by definition is the set of all points of the form

$$(1 - t)x + ty, \qquad \text{where } \; t \in \mathbb{R}.$$

Here's the picture:



This picture really is more than just schematic, as the line is basically a 1-dimensional object, even though it is located as a subset of n-dimensional space. In addition, the closed line *segment* with end points x and y consists of all points as above, but with $0 \le t \le 1$. This segment is shown above in heavier ink. We denote this segment by $[x, y]$.

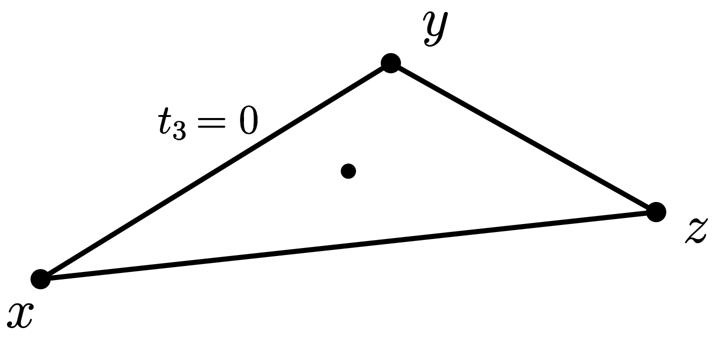
We now see right away the wonderful interplay between *algebra* and *geometry*, something that will occur frequently in this book. Namely, the points on the above line can be described completely in terms of the *algebraic* formula given for the line. On the other hand, the line is of course a *geometric* object.

It is very important to get comfortable with this sort of interplay. For instance, if we happen to be discussing points in $\mathbb{R}^5$, we probably have very little in our background that gives us geometric insight to the nature of $\mathbb{R}^5$. However, the algebra for a line in $\mathbb{R}^5$ is very simple, and the geometry of a line is just like the geometry of $\mathbb{R}^1$.

Similarly, it is helpful to represent *triangles* with a picture in the plane of the page. Thus if we have three noncollinear points x, y, z in $\mathbb{R}^n$, there is a unique *plane* which contains them. This plane lies in $\mathbb{R}^n$ of course, but restricting attention to it gives a picture that looks like an ordinary plane. The plane is the set of all points of the form

$$p = t_1 x + t_2 y + t_3 z,$$

where $t_1 + t_2 + t_3 = 1$. Sometimes the scalars t_1, t_2, t_3 are called *barycentric* coordinates of the point p.



The point displayed inside this triangle is $\frac{1}{3}(x+y+z)$, and is called the *centroid* of the triangle.

PROBLEM 1–1. We need to examine the word *collinear* we have just used. In fact, prove that three points x, y, z in $\mathbb{R}^n$ lie on a line $\iff$ there exist scalars t_1, t_2, t_3, not all zero, such that

$$\begin{aligned} t_1 + t_2 + t_3 &= 0, \\ t_1 x + t_2 y + t_3 z &= 0. \end{aligned}$$

PROBLEM 1–2. Prove that x, y, 0 are collinear $\iff$ x is a scalar multiple of y or y is a scalar multiple of x.

PROBLEM 1–3. Prove that if x, y, z in $\mathbb{R}^n$ are not collinear and if p belongs to the plane they determine, then the real numbers t_1, t_2, t_3 such that

$$\begin{aligned} t_1 + t_2 + t_3 &= 1, \\ t_1 x + t_2 y + t_3 z &= p, \end{aligned}$$

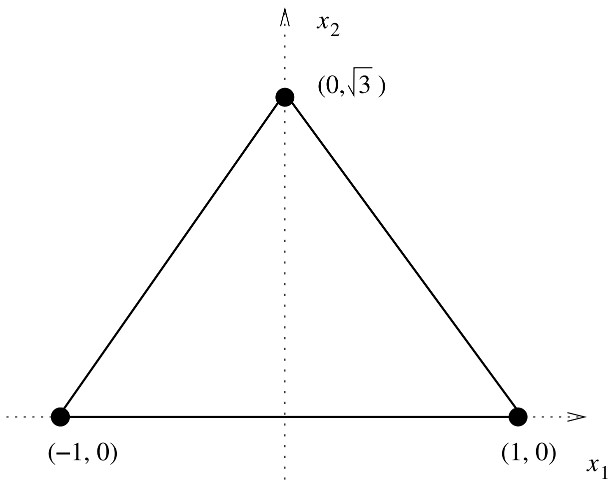
are unique.

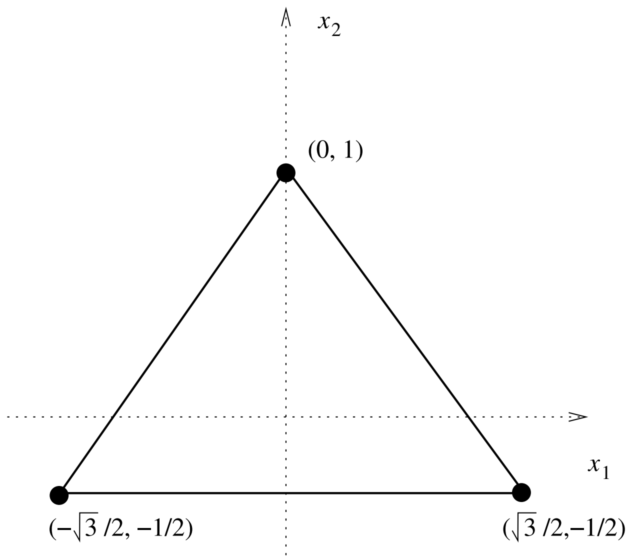
PROBLEM 1–4. In the triangle depicted above let L_1 be the line determined by x and the midpoint $\frac{1}{2}(y+z)$, and L_2 the line determined by y and the midpoint $\frac{1}{2}(x+z)$. Show that the intersection $L_1 \cap L_2$ of these lines is the centroid. (This proves the theorem which states that *the medians of a triangle are concurrent.*)

PROBLEM 1–5. Prove that the *interior* (excluding the sides) of the above triangle is described by the conditions on the barycentric coordinates

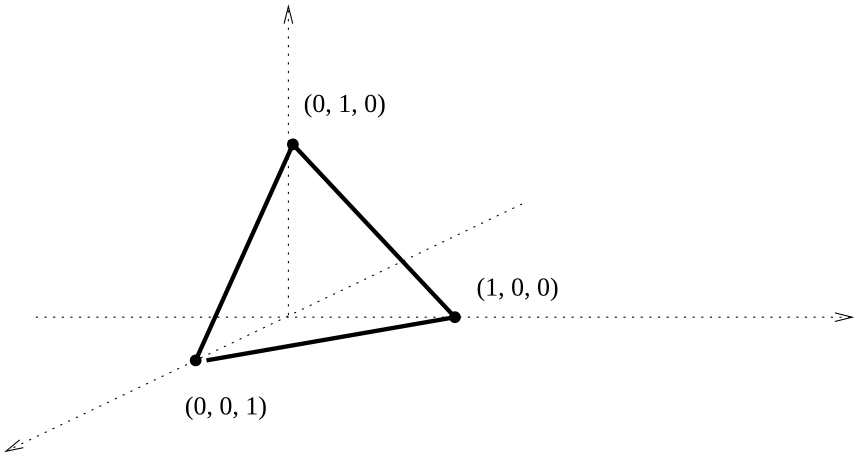
$$t_1 + t_2 + t_3 = 1,$$
$$t_1 > 0, t_2 > 0, \quad \text{and} \quad t_3 > 0.$$

As an example of our method of viewing triangles, think about an equilateral triangle. If we imagine it conveniently placed in $\mathbb{R}^2$, the coordinates of the vertices are bound to be rather complicated; for instance, here are two ways:

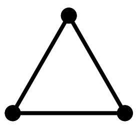




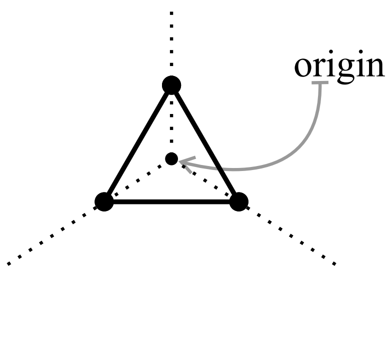
But a really elegant positioning is available in $\mathbb{R}^3$, if we simply place the vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$:



Now this looks much better if we view this triangle as it lies in the *plane* $x_1 + x_2 + x_3 = 1$:



We can't draw coordinate axes in this plane, or even the origin, though we could imagine the origin as sitting "behind" the centroid:

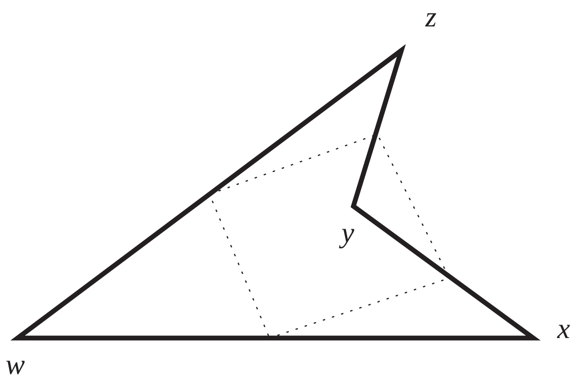


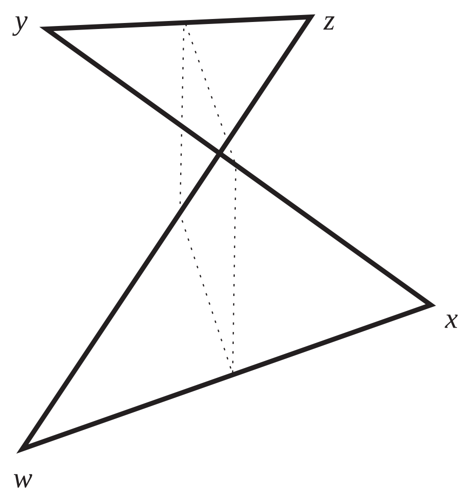
PROBLEM 1–6. Prove that the four points w, x, y, z in $\mathbb{R}^n$ are *coplanar* $\iff$ there exist real numbers t_1, t_2, t_3, t_4, not all zero, such that

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= 0, \\ t_1 w + t_2 x + t_3 y + t_4 z &= 0. \end{aligned}$$

As further evidence of the power of vector algebra in solving simple problems in geometry, we offer

PROBLEM 1–7. Let us say that four distinct points w, x, y, z in $\mathbb{R}^n$ define a *quadrilateral*, whose *sides* are the segments $[w, x]$, $[x, y]$, $[y, z]$, $[z, w]$ in that order.





Prove that the four midpoints of the sides of a quadrilateral, taken in order, form the vertices of a *parallelogram* (which might be degenerate). In particular, these four points are coplanar. (Note that the original quadrilateral need not lie in a plane.) Express the center of this parallelogram in terms of the points w, x, y, and z.

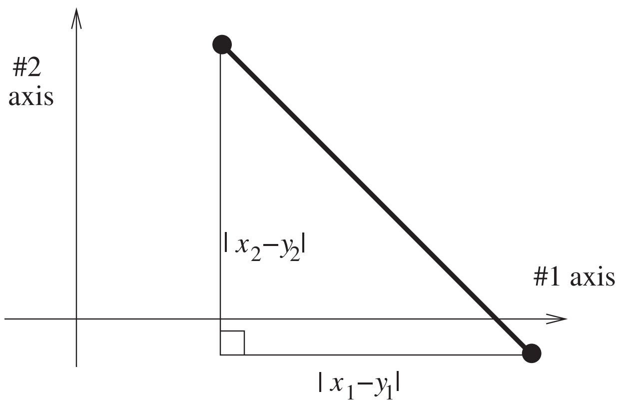
PROBLEM 1–8. We have seen in Problem 1–1 the idea of three points being collinear, and in Problem 1–6 the idea of four points being coplanar. These definitions can of course be generalized to an arbitrary number of points. In particular, give the correct analogous definition for two points to be (say) "copunctual" and then prove the easy result that two points are "copunctual" if and only if they are equal.

PROBLEM 1–9. Give a careful proof that any three points in $\mathbb{R}^1$ are collinear; and also that any four points in $\mathbb{R}^2$ are coplanar.

It is certainly worth a comment that we expect three "random" points in $\mathbb{R}^2$ to be noncollinear. We would then say that the three points are in *general position.* Likewise, four points in $\mathbb{R}^3$ are in *general position* if they are not coplanar.

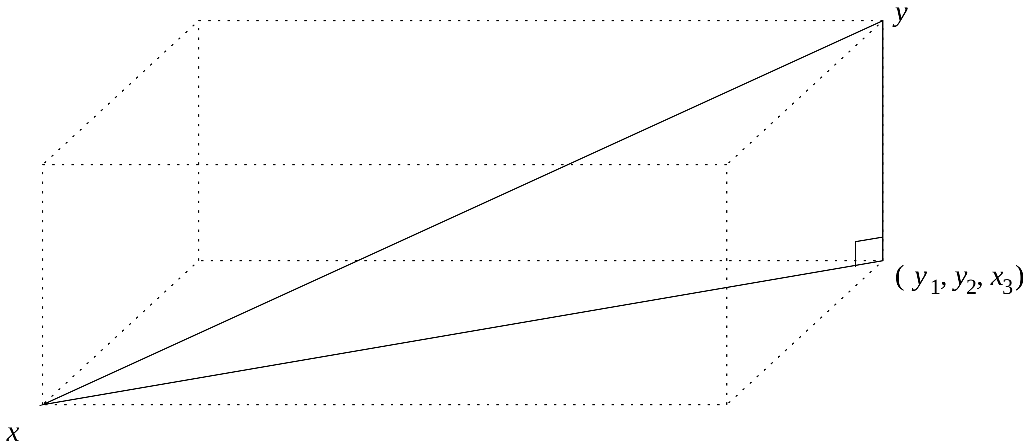
B. **Distance**

Now we are going to discuss the all-important notion of distance in $\mathbb{R}^n$. We start with $\mathbb{R}^2$, where we have the advantage of really understanding and liking the *Pythagorean theorem.* We shall freely accept and use facts we have learned from standard plane geometry. Thus we say that the *distance* between x and y in $\mathbb{R}^2$ is



$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We can even proceed easily to $\mathbb{R}^3$ by applying Pythagoras to the right triangle with legs given by the two segments: (1) from (x_1, x_2, x_3) to (y_1, y_2, x_3) and (2) from (y_1, y_2, x_3) to (y_1, y_2, y_3).



The segment (1) determines the distance

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

and the segment (2) determines the distance

$$|x_3 - y_3|.$$

If we square these distances, add the results, and then take the square root, the distance we find is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Though we are hard pressed to draw a similar picture for $\mathbb{R}^4$ etc., we can easily imagine the same procedure. For $\mathbb{R}^4$ we would consider the "horizontal" line segment from (x_1, x_2, x_3, x_4) to (y_1, y_2, y_3, x_4), and then the "vertical" segment from (y_1, y_2, y_3, x_4) to (y_1, y_2, y_3, y_4). The two corresponding distances are

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

and

$$|x_4 - y_4|,$$

respectively. (We have used our formula from the previous case of $\mathbb{R}^3$.) We certainly want to think of these two segments as the legs of a right triangle, so that the distance between x and

y should come from Pythagoras by squaring the two numbers above, adding, and then taking the square root:

$$d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+(x_4-y_4)^2}.$$

We don't really need a picture to imagine this sort of construction for any $\mathbb{R}^n$, so we are led to

THE PYTHAGOREAN DEFINITION. The *distance* between x and y in $\mathbb{R}^n$ is

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}.$$

Clearly, $d(x,y) = d(y,x)$, $d(x,x) = 0$, and $d(x,y) > 0$ if $x \neq y$. We also say that $d(x,y)$ is the *length* of the line segment $[x,y]$.

This definition of course gives the "right" answer for $n = 2$ and $n = 3$. (It even works for $n = 1$, where it decrees that $d(x,y) = \sqrt{(x-y)^2} = |x-y|$.)

It will be especially convenient to have a special notation for the distance from a point to the origin:

DEFINITION. The *norm* of a point x in $\mathbb{R}^n$ is the number

$$\|x\| = d(x,0) = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Thus we have equations like

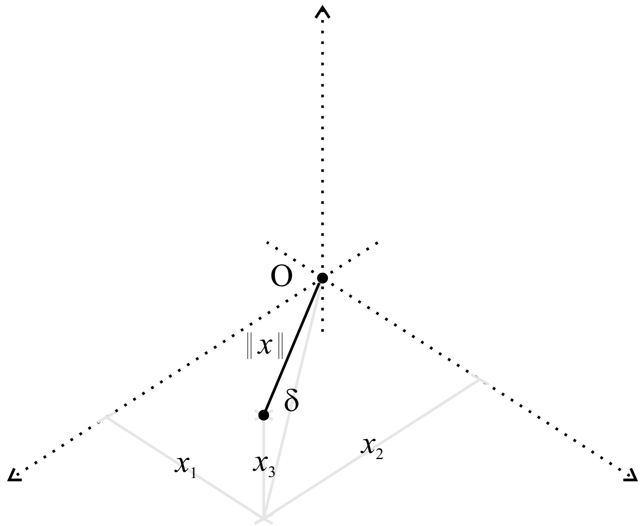
$$\begin{aligned}
\mathbb{R}^1 &: & \|-3\| &= 3\\
\mathbb{R}^2 &: & \|(3,-4)\| &= 5\\
\mathbb{R}^3 &: & \|(1,-2,3)\| &= \sqrt{14}\\
\mathbb{R}^4 &: & \|(1,1,1,1)\| &= 2.
\end{aligned}$$

The idea of "norm" is important in many areas of mathematics. The particular definition we have given is sometimes given the name *Euclidean* norm.

We have some easy properties:

$$\begin{aligned}
d(x,y) &= \|x-y\|;\\
\|ax\| &= |a|\ \|x\| \text{ for } a \in \mathbb{R};\\
\therefore \|-x\| &= \|x\|;\\
\|0\| &= 0;\\
\|x\| &> 0 \text{ if } x \neq 0.
\end{aligned}$$

Here's a typical picture in $\mathbb{R}^3$:



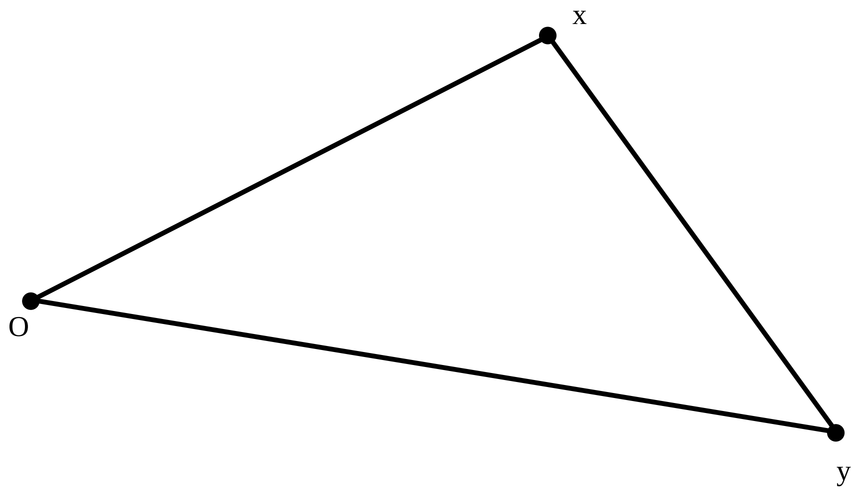
$$\begin{aligned} \|x\|^2 &= \delta^2 + x_3^2 \\ \delta^2 &= x_1^2 + x_2^2 \end{aligned}$$

We close this section with another "algebra-geometry" remark. We certainly are thinking of distance geometrically, relying heavily on our $\mathbb{R}^2$ intuition. Yet we can calculate distance algebraically, thanks to the formula for $d(x, y)$ in terms of the coordinates of the points x and y in $\mathbb{R}^n$.

C. **Right angle**

Now we turn to a discussion of orthogonality. We again take our clue from the Pythagorean theorem, *the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides*. The key word we want to understand is *right*.

Thus we want to examine a triangle in $\mathbb{R}^n$ (with $n \geq 2$). Using a translation, we may presume that the potential right angle is located at the origin. Thus we consider from the start a triangle with vertices 0, x, y. As we know that these points lie in a plane, it makes sense to think of them in a picture such as



We are thus looking at the plane containing 0, x, y, even though these three points lie in $\mathbb{R}^n$.

We then say that the angle at 0 is a right angle *if and only if* the Pythagorean identity holds:

$$d(x,y)^2 = d(x,0)^2 + d(y,0)^2.$$

I.e., in terms of the Euclidean norm on $\mathbb{R}^n$,

$$\|x-y\|^2 \;=\; \|x\|^2 + \|y\|^2. \tag{P}$$

You probably have noticed that our reasoning has been somewhat circular in nature. For we used the Pythagorean idea to motivate the definition of distance in the first place, and now we are using distance to define right angle. There is an important subtlety at work here. Namely, in defining distance we worked with right angles *in coordinate directions only*, whereas now we are defining right angles in arbitrary (noncoordinate) directions. Thus we have achieved something quite significant in this definition.

We feel justified in this definition because of the fact that it is based on our intuition from the Euclidean geometry of $\mathbb{R}^2$. Our planes are located in $\mathbb{R}^n$, but we want them to have the same geometric properties as $\mathbb{R}^2$.

Now we perform a calculation based upon our use of algebra in this material:

$$\begin{aligned}\|x-y\|^2 &= \sum_{i=1}^{n}(x_i-y_i)^2\\ &= \sum_{i=1}^{n}(x_i^2-2x_iy_i+y_i^2)\\ &= \|x\|^2-2\sum_{i=1}^{n}x_iy_i+\|y\|^2.\end{aligned}$$

Thus the Pythagorean relation (P) becomes

$$\sum_{i=1}^{n}x_iy_i=0.$$

In summary, we have a right angle at 0 $\iff \sum_{i=1}^{n}x_iy_i=0$.

Based upon the sudden appearance of the above number, we now introduce an extremely useful bit of notation:

DEFINITION. For any x, $y\in\mathbb{R}^n$, the *inner product* of x and y, also known as the *dot product*, is the number

$$\boxed{x\bullet y=\sum_{i=1}^{n}x_iy_i.}$$

I like to make this dot huge!

The above calculation thus says that

$$\|x-y\|^2 = \|x\|^2-2x\bullet y+\|y\|^2.$$

Just to make sure we have the definition down, we rephrase our definition of right angle:

DEFINITION. If x, $y\in\mathbb{R}^n$, then x and y are *orthogonal* (or *perpendicular*) if $x\bullet y=0$.

Inner product algebra is very easy and intuitive:

$$\begin{aligned}x\bullet y &= y\bullet x;\\ (x+y)\bullet z &= x\bullet z+y\bullet z;\\ (ax)\bullet y &= a\,(x\bullet y)\,;\\ 0\bullet x &= 0;\\ x\bullet x &= \|x\|^2.\end{aligned}$$

The calculation we have performed can now be done completely formally:

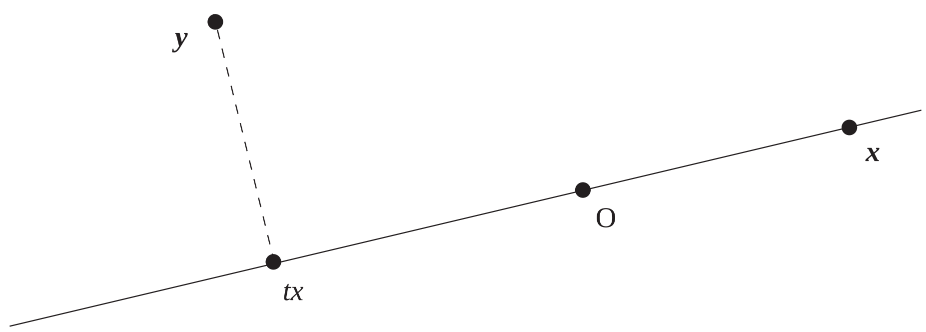
$$\begin{aligned}\|x-y\|^2 &= (x-y)\bullet(x-y)\\ &= x\bullet x - x\bullet y - y\bullet x + y\bullet y\\ &= \|x\|^2 - 2x\bullet y + \|y\|^2.\end{aligned}$$

PROBLEM 1–10. As we have noted, the vector 0 is orthogonal to every vector. Show conversely that if $x \in \mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$, then $x = 0$.

PROBLEM 1–11. Given $x, y \in \mathbb{R}^n$. Prove that $x = y \iff x\bullet z = y\bullet z$ for all $z \in \mathbb{R}^n$.

Here is an easy but astonishingly important

PROBLEM 1–12. Let $x \neq 0$ and y be in $\mathbb{R}^n$. As we know, the line determined by 0 and x consists of all points of the form tx. Find the (unique) point on this line such that the vector $y - tx$ is orthogonal to x. Also calculate as elegantly as you can the distance $\|y - tx\|$.



PROBLEM 1–13. In the same situation, find the (unique) point on the line which is closest to y. Comment?

(Solutions: $(y-tx)\bullet x = 0 \iff y\bullet x - t\|x\|^2 = 0 \iff t = x\bullet y/\|x\|^2$. Also we have

$$\|y-tx\|^2 = \|y\|^2 - 2tx\bullet y + t^2\|x\|^2.$$

For the above value of t, we obtain

$$\begin{aligned}\|y - tx\|^2 &= \|y\|^2 - \frac{2x \bullet y}{\|x\|^2} x \bullet y + \frac{(x \bullet y)^2}{\|x\|^2} \\ &= \|y\|^2 - \frac{(x \bullet y)^2}{\|x\|^2}.\end{aligned}$$

Thus

$$\|y - tx\| = \frac{\sqrt{\|x\|^2\|y\|^2 - (x \bullet y)^2}}{\|x\|}.$$

This is the solution of 1–12. To do 1–13 write the formula above in the form

$$\begin{aligned}\|y - tx\|^2 &= \|x\|^2 \left[t^2 - 2t\frac{x \bullet y}{\|x\|^2}\right] + \|y\|^2 \\ &= \|x\|^2 \left(t - \frac{x \bullet y}{\|x\|^2}\right)^2 - \frac{(x \bullet y)^2}{\|x\|^2} + \|y\|^2.\end{aligned}$$

The minimum occurs $\iff t = x \bullet y / \|x\|^2$. The *comment* is that the same point is the solution of both problems.)

Another way to handle Problem 1–13 is to use calculus to find the value of t which minimizes the quadratic expression.

You should trust your geometric intuition to cause you to believe strongly that the point asked for in Problem 1–12 must be the same as that asked for in Problem 1–13.

PROBLEM 1–14. This problem is a special case of a two-dimensional version of the preceding two problems. Let $n \geq 2$ and let M be the subset of $\mathbb{R}^n$ consisting of all points of the form $x = (x_1, x_2, 0, \dots, 0)$. (In other words, M is the $x_1 - x_2$ plane.) Let $y \in \mathbb{R}^n$.

a. Find the unique $x \in M$ such that $y - x$ is orthogonal to all points in M.

b. Find the unique $x \in M$ which is closest to y.

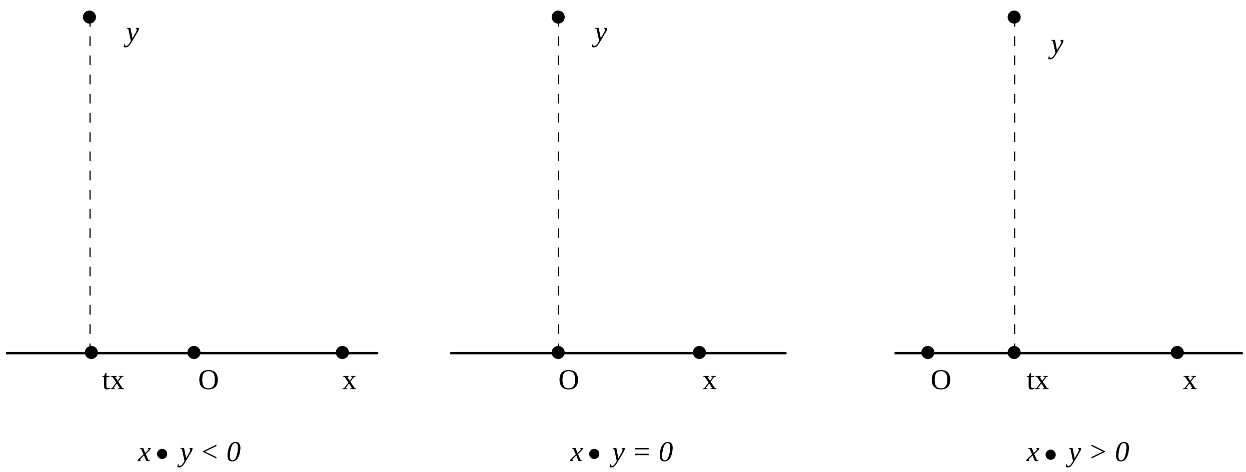
BONUS. Since $\|y - tx\|^2 \geq 0$, we conclude from the above algebra that $\|x\|^2\|y\|^2 - (x \bullet y)^2 \geq 0$. Furthermore, this can be equality $\iff y - tx = 0 \iff y = tx$. Now that we have proved this, we state it as the

SCHWARZ INEQUALITY. *For any x, $y \in \mathbb{R}^n$,*

$$\boxed{|x \bullet y| \leq \|x\| \; \|y\|.}$$

Furthermore, equality holds $\iff x = 0$ *or* $y = 0$ *or* y *is a scalar multiple of* x.

It is extremely useful to keep in mind schematic figures to illustrate the geometric significance of the *sign* of $x \bullet y$:

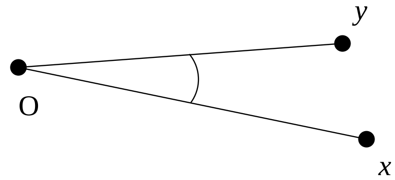


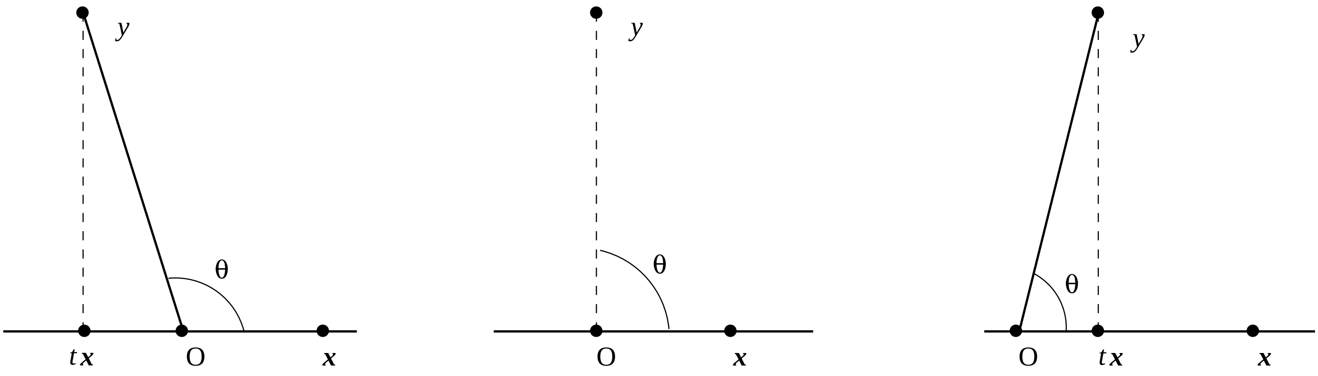
The validity of these figures follows from the formula we have obtained for t, which implies that t and $x \bullet y$ have the same sign.

D. **Angles**

Amazingly, we can now use our understanding of *right* angle to define measurement of angles in general. Suppose x and y are nonzero points in $\mathbb{R}^n$ which are not scalar multiples of each other. In other words, x, y, and 0 are not collinear. We now scrutinize the plane in $\mathbb{R}^n$ which contains x, y, and 0. Specifically, we examine the angle formed at 0 by the line segments from 0 to x and from 0 to y, respectively.

To measure this angle we use the most elementary definition of *cosine* from high school geometry. We have the three cases depicted at the end of Section C:

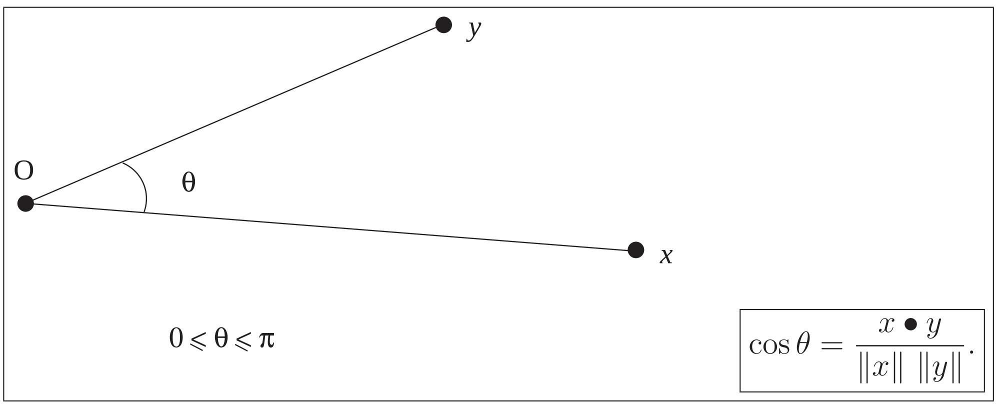




In all cases we let θ be the angle between the two segments we are considering. Then

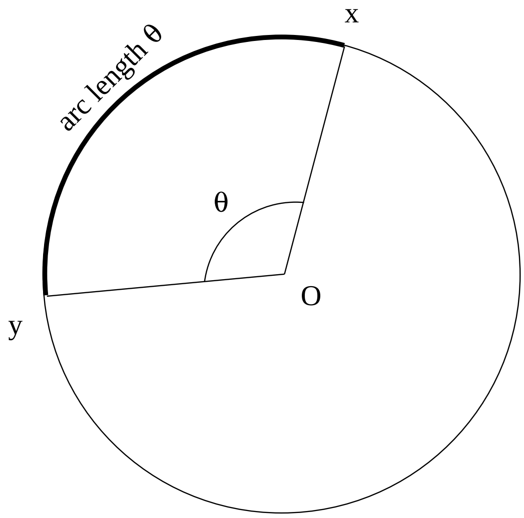
$$\cos\theta = \frac{\text{“adjacent side”}}{\text{“hypotenuse”}} = \frac{t\|x\|}{\|y\|}.$$

Remember that $t = x \bullet y/\|x\|^2$. Therefore we have our desired formula for θ:



At the risk of excessive repetition, this sketch of the geometric situation is absolutely accurate, as we are looking at a *plane* contained in $\mathbb{R}^n$.

Once again the interplay between algebra and geometry is displayed. For the definition of the inner product $x \bullet y$ is given as an algebraic expression in the coordinates of the two vectors, whereas now we see $x \bullet y$ is intimately tied to the geometric idea of angle. What is more, in case x and y are *unit* vectors (meaning that their norms equal 1), then $\cos\theta = x \bullet y$. A nice picture for this is obtained by drawing a circle of radius 1, centered at 0, in the plane of 0, x, and y. Then x and y lie on this circle, and θ is the length of the shorter arc connecting x and y.

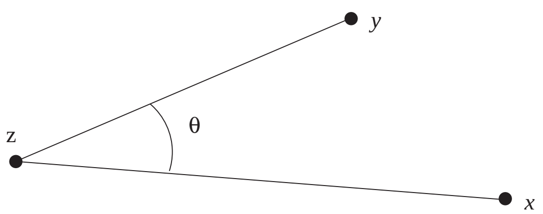


SUMMARY. The inner product on $\mathbb{R}^n$ is a wonderful two-edged sword. First, $x \bullet y$ is a completely geometric quantity, as it equals the product of the lengths of the two factors and the cosine of the angle between them. Second, it is easily computed algebraically in terms of the coordinates of the two factors.

Any nonzero vector x produces a unit vector by means of the device of "dividing out" the norm: $x/\|x\|$. Then the above formula can be rewritten

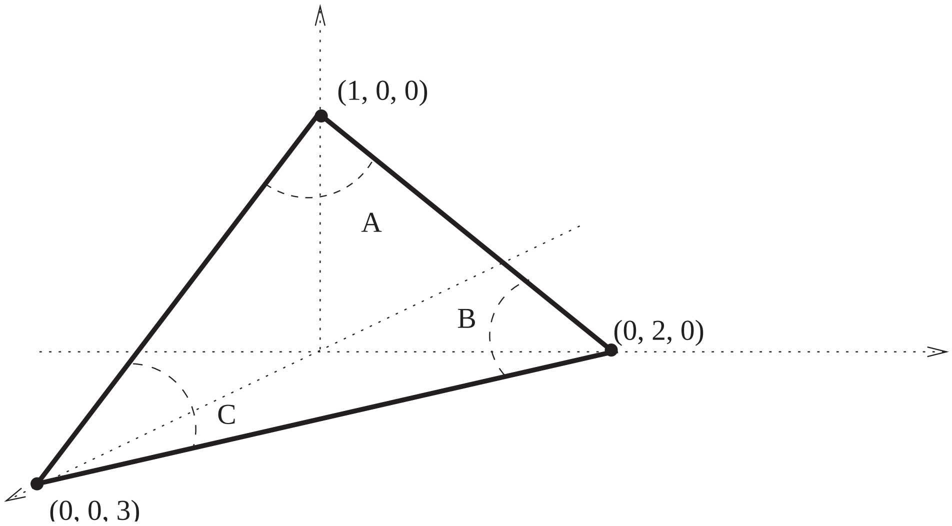
$$\boxed{\cos\theta = \frac{x}{\|x\|} \bullet \frac{y}{\|y\|}.}$$

REMARK. If the vertex of an angle is not the origin, then subtraction of points gives the correct formula:



$$\cos\theta = \frac{(x-z) \bullet (y-z)}{\|x-z\| \; \|y-z\|}.$$

EXAMPLE. A triangle in $\mathbb{R}^3$ has vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. What are its three angles? Solution:



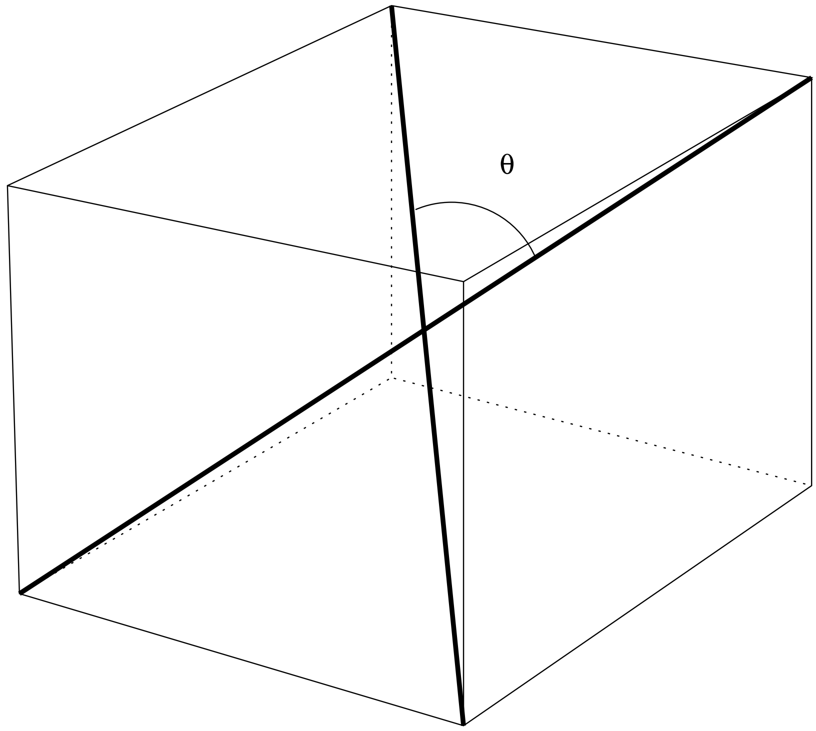
$$\begin{aligned}
\cos A &= \frac{(-1,2,0) \bullet (-1,0,3)}{\sqrt{5} \quad\quad \sqrt{10}} = \frac{1}{\sqrt{50}}. \\
\cos B &= \frac{4}{\sqrt{65}}, \\
\cos C &= \frac{9}{\sqrt{130}}.
\end{aligned}$$

Approximately,

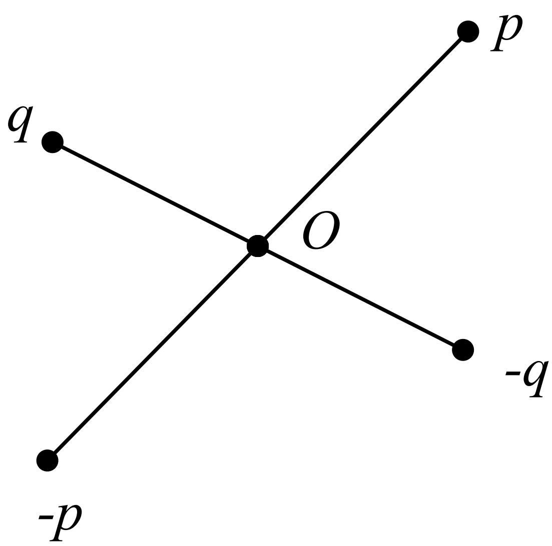
$$\begin{aligned}
A &= 81.87^{\text{o}}\,, \\
B &= 60.26^{\text{o}}\,, \\
C &= 37.87^{\text{o}}\,.
\end{aligned}$$

Notice how *algebraic* this is! And yet it produces valuable geometric information as to the shape of the given triangle.

EXAMPLE. Compute the acute angle between the diagonals of a cube in $\mathbb{R}^3$.



Solution: it is enough to arrange the cube so that its eight vertices are located at the points $(\pm 1, \pm 1, \pm 1)$. Then the diagonals are the line segments from one vertex p to the opposite vertex $-p$. These diagonals intersect at 0. The picture in the plane determined by two diagonals looks like this:



We are supposed to find the *acute* angle of their intersection, so we use

$$\cos\theta = \frac{p \bullet q}{\|p\| \; \|q\|}.$$

(If this turns out to be negative, as in the sketch, then we have $\frac{\pi}{2} < \theta < \pi$ and we use $\pi - \theta$ for the answer.) As $p = (\pm 1, \pm 1, \pm 1)$ and $q = (\pm 1, \pm 1, \pm 1)$, and $p \neq \pm q$, we have $\|p\| = \|q\| = \sqrt{3}$ and $p \bullet q = \pm 1$. Thus

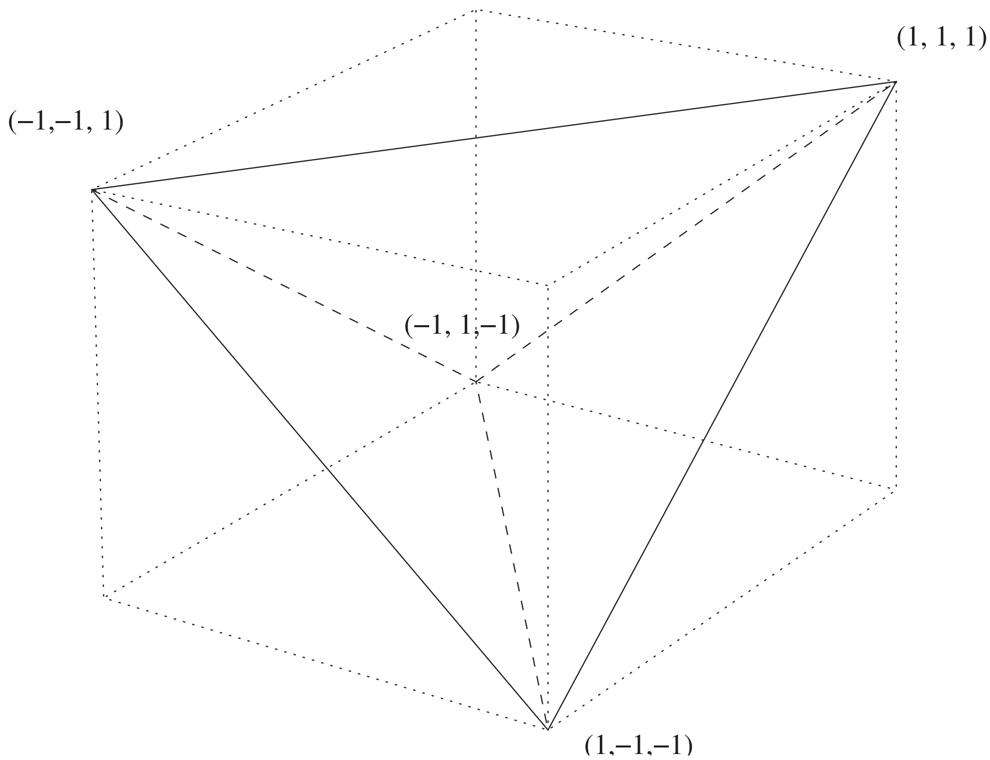
$$\cos\theta = \frac{1}{3},$$

giving

$$\theta = \arccos\frac{1}{3} \qquad (\cong 70.5^\circ).$$

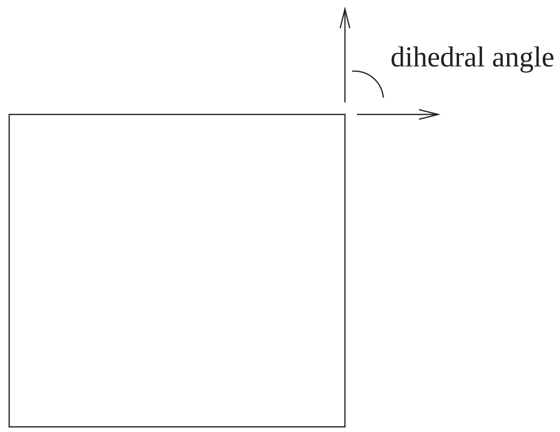
PROBLEM 1–15. Consider two diagonals of *faces* of a cube which intersect at a vertex of the cube. Compute the angle between them.

PROBLEM 1–16. Given a regular tetrahedron (its four faces are equilateral triangles), locate its centroid. (You may define its centroid to be the average of its four vertices; in other words, the (vector) sum of the vertices divided by 4.) Then consider two line segments from the centroid to two of the four vertices. Calculate the angle they form at the center. (Here is displayed a particularly convenient location of a regular tetrahedron.)



PROBLEM 1–17. Repeat the calculation of the receding problem but instead use a regular tetrahedron situated in $\mathbb{R}^4$ having vertices at the four unit coordinate vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$.

PROBLEM 1–18. Two adjacent faces of a cube intersect in an edge and form a *dihedral* angle, which is clearly $\frac{\pi}{2}$:



Side view of cube

Calculate the dihedral angle formed by two faces of a regular tetrahedron.

An important consequence of the Schwarz inequality is the

TRIANGLE INEQUALITY. *For any* x, $y \in \mathbb{R}^n$,

$$\|x + y\| \le \|x\| + \|y\|.$$

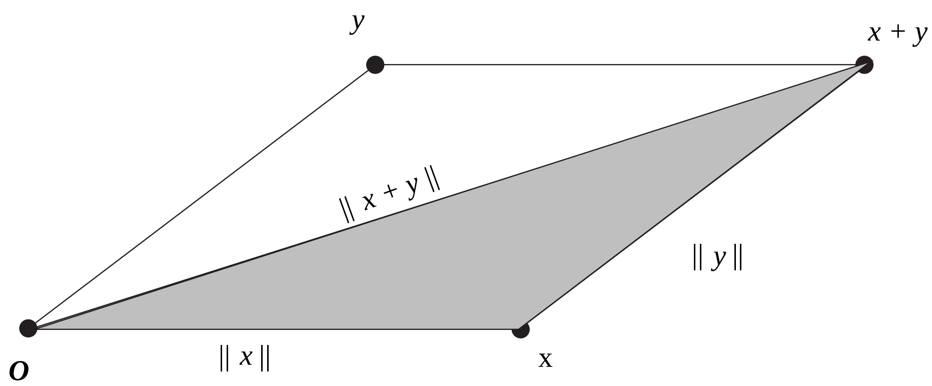
PROOF. We simply compute as follows:

$$\begin{aligned} \|x+y\|^2 &= (x+y) \bullet (x+y) \\ &= \|x\|^2 + 2x \bullet y + \|y\|^2 \\ &\le \|x\|^2 + 2|x \bullet y| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \qquad \text{(Schwarz inequality)} \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

QED

The reason for the name "triangle inequality" can be seen in a picture:

The shaded triangle has edges with lengths as shown, so the triangle inequality is the statement that *any edge of a triangle in* $\mathbb{R}^n$ *is less than the sum of the other two edges*.



PROBLEM 1–19. Prove that the triangle inequality is an equality $\iff$ either $x = 0$ or $y = tx$ for some $t \geq 0$. What does this mean geometrically?

PROBLEM 1–20. Use the triangle inequality to prove that for any points x, y, $z \in \mathbb{R}^n$,

$$d(x, y) \leq d(x, z) + d(z, y).$$

And prove that equality holds $\iff$ z belongs to the line segment $[x, y]$.

PROBLEM 1–21. Prove that for any x, $y \in \mathbb{R}^n$,

$$|\ \|x\| - \|y\|\ | \leq \|x - y\|.$$

Also prove that for any x, y, $z \in \mathbb{R}^n$,

$$|\ d(x, y) - d(x, z)\ | \leq d(y, z).$$

We should pause to wonder why it should be necessary to prove the triangle inequality, as we know this inequality from elementary plane geometry. The reason is twofold: First, we are after all working in $\mathbb{R}^n$ and this requires us to take great care lest we make an unwarranted assumption. Second, it is a truly wonderful accomplishment to be able to prove our results with such minimal assumptions; this can enable us to conclude similar results under circumstances which seem at first glance to be quite different.

PROBLEM 1–22. Prove the **PARALLELOGRAM "LAW"**: *the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its edges.* (HINT: explain why this is equivalent to

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.)$$

Before proceeding to trigonometry, a few words might be said about some generalizations of the above material. There are certainly many other ways of measuring distances on $\mathbb{R}^n$. For instance, we might sometimes want to use the norm

$$\|x\|_\infty = \max_{i=1,\dots,n} |x_i|.$$

This quantity is called a norm because it satisfies the basic properties

$$\begin{aligned} \|ax\|_\infty &= |a|\ \|x\|_\infty, \\ \|0\|_\infty &= 0, \\ \|x\|_\infty &> 0 \quad \text{if} \quad x \neq 0, \\ \|x+y\|_\infty &\leq \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

Another choice of norm even has its own special name, the *taxicab* norm:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

It too satisfies the listed properties.

Neither of these two norms arises from an inner product *via* a formula

$$\|x\| = \sqrt{x \bullet x}.$$

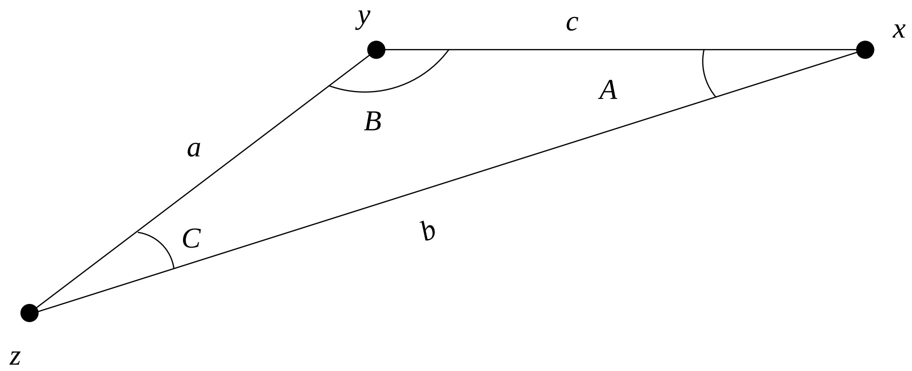
But other norms do so arise. For instance, define an inner product on $\mathbb{R}^3$ by means of the expression

$$x \star y = x_1y_1 + 3x_2y_2 + 10x_3y_3.$$

Then $\sqrt{x \star x}$ gives a norm on $\mathbb{R}^3$.

E. **A little trigonometry, or, we can now recover all the Euclidean geometry of triangles**

We now have more than enough information to be able to discuss triangles in $\mathbb{R}^n$ rather well. So consider an arbitrary triangle located in $\mathbb{R}^n$ ($n \geq 2$), with vertices, edges, and interior angles as shown:



Then we have immediately

$$\begin{aligned} c^2 &= \|x - y\|^2 \\ &= \|(x - z) - (y - z)\|^2 \\ &= \|x - z\|^2 - 2(x - z) \bullet (y - z) + \|y - z\|^2 \\ &= \|x - z\|^2 - 2\|x - z\| \, \|y - z\| \cos C + \|y - z\|^2 \\ &= b^2 - 2ba \cos C + a^2. \end{aligned}$$

This so-called "law" is the famous

$$\boxed{\text{LAW OF COSINES} \quad \Big| \quad c^2 = a^2 + b^2 - 2ab \cos C}.$$

Notice how elegantly this result is a consequence of easy algebra.

Next we employ the ordinary high school definition of the area of a triangle:

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}.$$

Thus,

$$\text{area} = \frac{1}{2} \times b \times a \sin C.$$

This implies

$$\frac{\sin C}{c} = \frac{2 \times \text{area}}{abc}.$$

As the right side is symmetric with respect to a, b, c, we obtain the

$$\boxed{\text{LAW OF SINES} \quad \Big| \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}.$$

Easy as it was to obtain the law of sines, there's an interesting different proof that uses essentially only algebra (not areas at all): use the law of cosines to calculate

$$\begin{aligned}
\sin^2 C &= 1 - \cos^2 C \\
&= (1 + \cos C)(1 - \cos C) \\
&= \left(1 + \frac{a^2 + b^2 - c^2}{2ab}\right)\left(1 - \frac{a^2 + b^2 - c^2}{2ab}\right) \\
&= \frac{2ab + a^2 + b^2 - c^2}{2ab} \times \frac{2ab - a^2 - b^2 + c^2}{2ab} \\
&= \frac{(a + b)^2 - c^2}{2ab} \times \frac{c^2 - (a - b)^2}{2ab} \\
&= \frac{(a + b + c)(a + b - c)}{2ab} \times \frac{(c + a - b)(c - a + b)}{2ab}.
\end{aligned}$$

Rearrange this to obtain

$$\sin C = \frac{\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}}{2ab}.$$

Again, we see that $\frac{\sin C}{c}$ is symmetric in a, b, c, yielding the law of sines once more.

There is a nice bonus. To state it in its classical form, we define the *semiperimeter* of the triangle to be

$$s = \frac{a + b + c}{2}.$$

Then

$$\begin{aligned}
\sin C &= \frac{\sqrt{2s(2s - 2a)(2s - 2b)(2s - 2c)}}{2ab} \\
&= 2\frac{\sqrt{s(s - a)(s - b)(s - c)}}{ab}.
\end{aligned}$$

By combining this result with our formula for $\sin C$ in terms of the area, we obtain the formula for the area of a triangle,

$$\boxed{\text{HERON'S FORMULA} \quad \Big| \quad \text{Area} = \sqrt{s(s - a)(s - b)(s - c)}}.$$

PROBLEM 1–23. Refer to the picture of the triangle at the beginning of this section. Prove that the angle bisectors of the three angles of the triangle are concurrent, intersecting at the point

$$\frac{ax+by+cz}{a+b+c}.$$

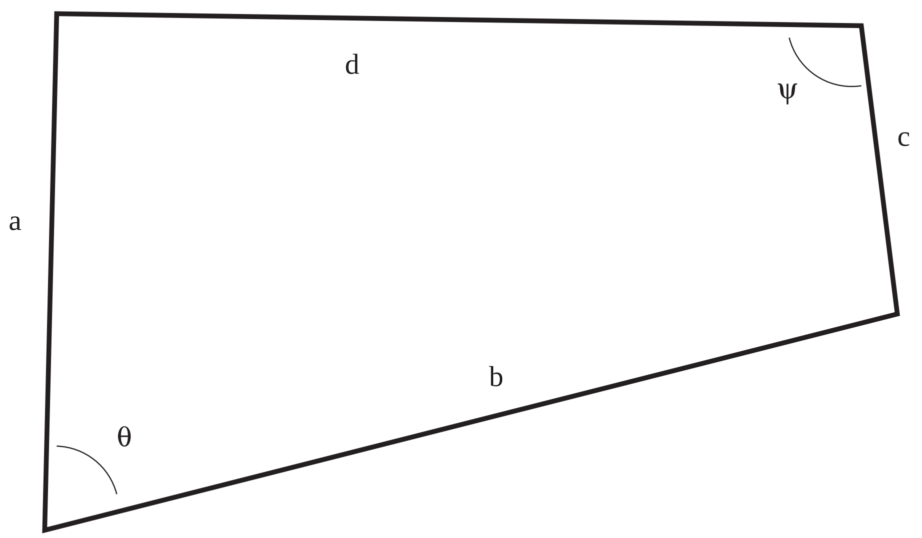
PROBLEM 1–24. Again consider a triangle in $\mathbb{R}^2$ with vertices x, y, z. We have considered its *centroid* $\frac{1}{3}(x+y+z)$ in Problem 1–4 and noted that it is the common intersection of the medians. This triangle has a unique *circumcircle*, the circle which passes through all three vertices. Its center q is called the *circumcenter* of the triangle. Prove that the point $p = x+y+z-2q$ is the *orthocenter*, the common intersection of the altitudes. (HINT: show that $(p-x) \bullet (y-z) = 0$, etc.)

PROBLEM 1–25. Prove that the centroid, circumcenter, and orthocenter of a triangle lie on a common line (known as the *Euler line*).

PROBLEM 1–26*. A *cyclic quadrilateral* is one whose vertices lie on a circle. Suppose that a *convex* cyclic quadrilateral is given, whose sides have lengths a, b, c, d, and whose semiperimeter is denoted $p = \frac{1}{2}(a+b+c+d)$. Prove this formula for its area:

BRAHMAGUPTA'S FORMULA	$\text{Area} = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

PROBLEM 1–27*. There is a result that greatly generalizes Brahmagupta's formula. Prove that for a general quadrilateral in $\mathbb{R}^2$ as shown, its area equals



$$\sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2 \tfrac{1}{2}(\theta+\psi)}.$$

(HINT: $\cos^2 \frac{1}{2}(\theta+\psi) = \frac{1}{2}(1+\cos(\theta+\psi))$

$$= \frac{1}{2}(1+\cos\theta\cos\psi - \sin\theta\sin\psi);$$

$\text{area} = \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\psi$ (why?);

$a^2+b^2-2ab\cos\theta = c^2+d^2-2cd\cos\psi$ (why?). Now have courage!)

PROBLEM 1–28. Prove that Heron's formula is an easy special case of Brahmagupta's.

PROBLEM 1–29. Show that the area of a general convex quadrilateral in $\mathbb{R}^2$ cannot be expressed as a function of only the lengths of the four sides.

PROBLEM 1–30. Given a convex quadrilateral in $\mathbb{R}^2$ whose area is given by Brahmagupta's formula, prove that it is necessarily cyclic.

F. **Balls and spheres**

Before developing the ideas of this section, we mention that mathematical terminology is in disagreement with the English language in denoting the basic objects. From *The American College Dictionary*, Harper 1948, we have:

ball 1. a spherical . . . body; a sphere

sphere 2. a globular mass, shell, etc. Syn. ball

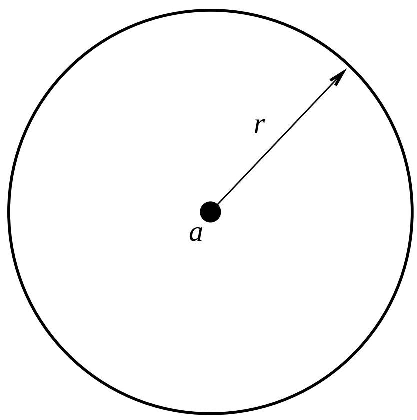
However, mathematics *always* uses "ball" to represent a "solid" object and "sphere" to represent its "surface." Here are the precise definitions.

Let $a \in \mathbb{R}^n$ and $0 < r < \infty$ be fixed. Then we define

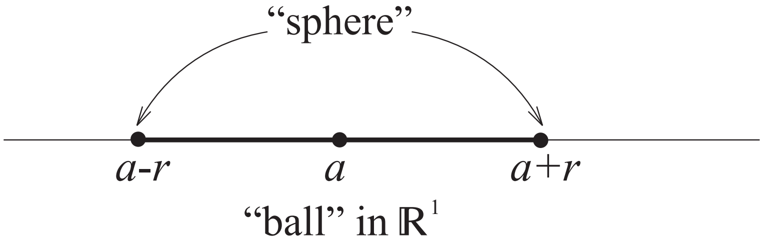
$B(a,r) = \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$ OPEN BALL WITH CENTER a AND RADIUS r;

$\overline{B}(a,r) = \{x \in \mathbb{R}^n \mid \|x-a\| \le r\}$ CLOSED BALL WITH CENTER a AND RADIUS r;

$S(a,r) = \{x \in \mathbb{R}^n \mid \|x-a\| = r\}$ SPHERE WITH CENTER a AND RADIUS r.



Of course in $\mathbb{R}^2$ we would use the words "disk" and "circle" instead of "ball" and "sphere," respectively. And in $\mathbb{R}^1$, $B(a,r)$ is the open "interval" from $a-r$ to $a+r$, and the "sphere" $S(a,r)$ consists of just the two points $a-r$ and $a+r$.



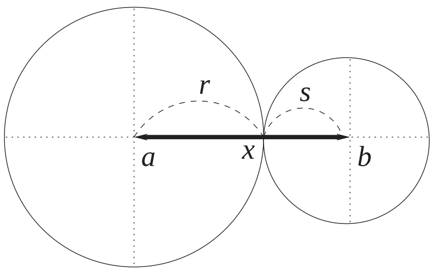
DEFINITION. If the center is 0 and the radius is 1, we call the above objects the *unit* open ball, the *unit* closed ball, and the *unit* sphere, respectively.

PROBLEM 1–31. Prove that the two closed balls $\overline{B}(a,r)$ and $\overline{B}(b,s)$ have a nonempty intersection $\iff \|a-b\| \le r+s$.

In the above problem the implication in the direction $\Rightarrow$ is straightforward: just use a point $x \in \overline{B}(a,r) \cap \overline{B}(b,s)$ and the triangle inequality. The opposite implication is subtler, as you must actually demonstrate the existence of some point in $\overline{B}(a,r) \cap \overline{B}(b,s)$. (The line segment $[a,b]$ is a good place to look.)

PROBLEM 1–32. Prove that the two open balls $B(a,r)$ and $B(b,s)$ have a nonempty intersection $\iff \|a-b\| < r+s$.

PROBLEM 1–33. Consider the preceding problems in the case of equality, $\|a-b\| = r+s$. Then the two balls are *tangent*, as in the figure. What is x equal to in terms of a, b, r, s?



PROBLEM 1–34. Prove that $\overline{B}(a,r) \subset \overline{B}(b,s) \iff$ a certain condition holds relating $\|a-b\|$ and r and s. This means you are first required to discover what the condition should be.

PROBLEM 1–35. Prove that for $n \geq 2$ the sphere $S(a,r)$ is quite "round," in the sense that there do not exist three distinct points in $S(a,r)$ which are collinear.
(HINT: WLOG (why?) assume you are working with the unit sphere. Assume x, y, z are collinear with z between x and y, so that $z = (1-t)x + ty$, where $0 < t < 1$, and assume $\|x\| = \|y\| = 1$ and $x \neq y$. Calculate $\|z\|^2$ and show $\|z\|^2 < 1$.)

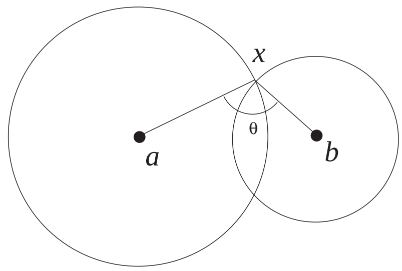
PROBLEM 1–36. Assume $n \geq 2$. Prove that the two spheres $S(a,r)$ and $S(b,s)$ have a nonempty intersection $\iff |r-s| \leq \|a-b\| \leq r+s$.
Explain why the intersection is like a sphere of radius R in $\mathbb{R}^{n-1}$, where

$$R^2 = \frac{r^2+s^2}{2} - \frac{\|a-b\|^2}{4} - \frac{(r^2-s^2)^2}{4\|a-b\|^2}.$$

(HINT: any point in $\mathbb{R}^n$ can be written uniquely in the form $(1-t)a + tb + u$, where $(a-b) \bullet u = 0$.)

PROBLEM 1–37. Assume $n \geq 2$. Given two intersecting spheres $S(a, r)$ and $S(b, s)$, say that the *angle* between them is the angle formed at a common point x by the vectors $a - x$ and $b - x$. Prove that this angle θ is independent of x, and satisfies

$$\cos\theta = \frac{r^2 + s^2 - \|a - b\|^2}{2rs}.$$



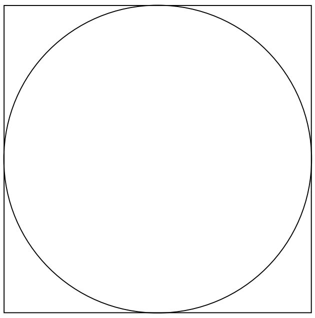
(Incidentally, notice that the inequalities in the statement of Problem 1–36 exactly state that $1 \geq \cos\theta \geq -1$, respectively.)

We are going to accomplish many wonderful things with balls and spheres in this course, including the completion of this interesting table:

dimension n	n-dimensional "volume" of $B(a, r)$	$(n-1)$-dimensional "volume" of $S(a, r)$
1	$2r$	2
2	πr^2	$2\pi r$
3	$\frac{4}{3}\pi r^3$	$4\pi r^2$
4	$\frac{1}{2}\pi^2 r^4$	$2\pi^2 r^3$
5	$\frac{8}{15}\pi^2 r^5$	$\frac{8}{3}\pi^2 r^4$
6	$\frac{1}{6}\pi^3 r^6$	$\pi^3 r^5$
$\vdots$	$\vdots$	$\vdots$

Probably you should be unable to see a pattern in going from n to $n + 1$ in this table, but you should definitely be able to see how to proceed from the volume of $B(a, r)$ to the volume of $S(a, r)$ for any given n. Do you see it?

It is fascinating to try to gain some intuition about $\mathbb{R}^n$ by meditating on balls. Some intuition indeed comes from the familiar cases of dimensions 1, 2, and 3, and *algebraically* it is relatively easy to understand the higher dimensional cases as well, at least on a superficial level. But *geometric* properties of balls and spheres in higher dimensions can appear bizarre at first glance. The following problems deal with the situation of a ball "inscribed" in a "cube" in $\mathbb{R}^n$. The picture in $\mathbb{R}^2$ appears as shown:



We begin these problems by assuming that the situation has been normalized so that the ball is the open unit ball $B(0,1)$ and the cube is the Cartesian product

$$C = [-1,1] \times \dots x[-1,1] = \{x \in \mathbb{R}^n \mid |x_i| \leq 1 \quad \text{for} \quad i = 1, \dots, n\}.$$

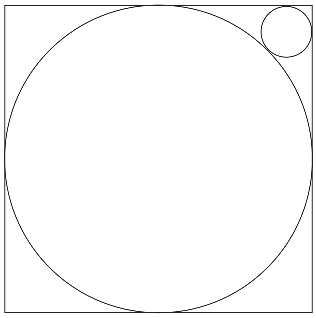
PROBLEM 1–38. Consider balls $B(a,r)$ which are contained in C and disjoint from $B(0,1)$.

a. Show that the maximum radius of such balls is $\frac{\sqrt{n}-1}{\sqrt{n}+1}$.

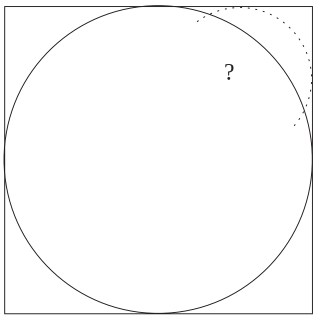
b. Show that the maximum radius is attained for precisely 2^n choices of the center a, namely

$$a = \frac{2}{\sqrt{n}+1}(\pm 1, \pm 1, \dots, \pm 1).$$

PROBLEM 1–39. Here is the picture in $\mathbb{R}^2$ of one of the four smaller balls:



With extreme accuracy and care, draw the corresponding picture in $\mathbb{R}^3$ as viewed looking straight at the $x_1 - x_2$ plane:



PROBLEM 1–40. Show that for the case of $\mathbb{R}^9$ each of the small balls is *tangent* to nine other of the small balls!

PROBLEM 1–41. Assume the fact which we shall prove later that the n-dimensional volume of any ball $B(a,r)$ in $\mathbb{R}^n$ has the form

$$vr^n,$$

where v is a constant (depending on the dimension n). Show that if $n = 9$ the total volume of the small balls equals the volume of $B(0,1)$, but this is not true for $n \neq 9$.

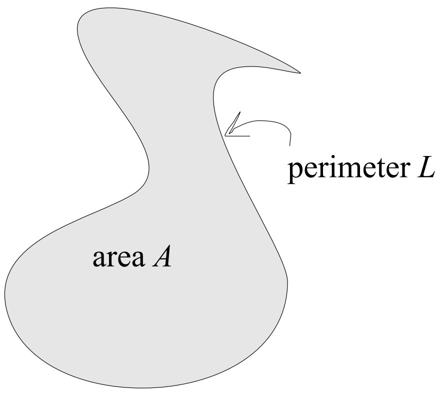
Incidentallly, the definition of C can be rephrased as follows: if we use the norm $\max_{i=1\dots n} |x_i|$ as on p. 1-18, then C is the closed unit "ball" with respect to that norm.

G. **Isoperimetric inequalities**

Thinking about areas of regions in $\mathbb{R}^2$ leads us to wonder about the following question: *of all plane regions of equal perimeter, which has the greatest area*? Everyone who thinks about this gives the same intuitive answer: a *disk*. And that is correct.

Though an actual proof of this innocent sounding statement is beyond our capability in this book, it is definitely of some interest to meditate on this problem, and to prove some interesting associated results.

First it is helpful to normalize the situation. Let us suppose we are dealing with a region of area A and perimeter L.



Then we want to compare A and L, but we want the comparison not to be confused with the actual size of the region. There are several ways to achieve this. One is to assume that L is fixed and then study how large A may be ("isoperimetric"). In doing this we may rescale to achieve for instance that $L = 1$. Another is to assume A is fixed, say $A = 1$, and study L.

These methods are clearly equivalent, and are also equivalent to studying the *dimensionless* ratio A/L^2. For whatever units of measurement we use, both numerator and denominator contain the square of the particular unit employed, and thus A/L^2 is truly dimensionless.

This means that we may multiply all points in $\mathbb{R}^2$ by any scalar $a > 0$, and note that the area of the region gets multiplied by a^2 and the perimeter by a. Thus A/L^2 is invariant.

For a disk of radius 1, $A = \pi$ and $L = 2\pi$, so that $A/L^2 = 1/4\pi$. So the *isoperimetric theorem* should say that for all regions in the plane

$$\frac{A}{L^2} \leq \frac{1}{4\pi}.$$

For instance, a square has $A/L^2 = 1/16$. A semidisk of radius 1 has $A = \pi/2$ and $L = \pi + 2$, so $A/L^2 \approx 0.06$.

A proof of the isoperimetric inequality is outside our interests for this book. However, in restricted situations we are able to obtain good results using familiar techniques. The following problem is a good instance.

PROBLEM 1–42. Prove that for any triangle in $\mathbb{R}^2$, the quotient A/L^2 is maximal if and only if the triangle is equilateral, using the following outline.

a. Show that Heron's formula requires proving that

$$\sqrt{(s-a)(s-b)(s-c)} \leq s^{3/2}/\sqrt{27}$$

with equality if and only if $a = b = c$.

b. Define $x = s - a$, $y = s - b$, $z = s - c$, and show that part a is equivalent to the classical inequality between geometric and arithmetic means (which you may assume),

$$(xyz)^{1/3} \leq \frac{x+y+z}{3}.$$

PROBLEM 1–43. Using the same technique, investigate triangles with perimeter 1 and a given value $0 < a < \frac{1}{2}$ for one of the sides. Show that the greatest area occurs when the triangle is isosceles with sides equal to a, $\frac{1-a}{2}$, $\frac{1-a}{2}$.

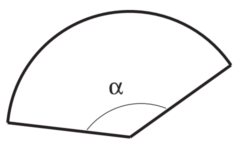
PROBLEM 1–44. Prove that for any quadrilateral in $\mathbb{R}^2$, the quotient A/L^2 is maximal if and only if the quadrilateral is a square.
(HINT: use Problem 1–27 and the same outline as in the preceding problems.)

PROBLEM 1–45. Consider a regular polygon in $\mathbb{R}^2$ with n sides. Show that its ratio is

$$\frac{A}{L^2} = \frac{\cot \frac{\pi}{n}}{4n}.$$

PROBLEM 1–46. As n increases, a regular n-gon increasingly resembles a disk. Prove that the ratios A/L^2 increase with n and have the limit $1/4\pi$ (the corresponding ratio for a disk).

PROBLEM 1–47. Among all sectors of a disk with opening α, which angle α produces the maximum value of A/L^2?



PROBLEM 1–48. Argue that the isoperimetric inequality for regions in $\mathbb{R}^3$ should have the form

$$\frac{\text{volume}}{(\text{surface area})^{3/2}} \leq \frac{1}{6\sqrt{\pi}}.$$

PROBLEM 1–49. Consider right-angled boxes in $\mathbb{R}^3$. Show that the ratio of the preceding problem is maximized only for cubes and that it is less than $1/6\sqrt{\pi}$.