ELEC 501 — **Serkan Gugercin**

Approximation Theory — November 8, 2002

Homework 3

Due Date: **November 22, 2002**

[1]. (*Slightly modified from [1]*) Given $\mathbf{\Sigma} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $\mathcal{P} \geq 0$ be the solution to the Lyapunov equation $A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, and A is asymptotically stable, i.e., $\Re(\lambda_i(A)) < 0$ for $i = 1, \cdots, n$. Suppose that $\mathcal{P}$ has q zero eigenvalues, i.e., $\mathcal{P}$ is positive semi-definite. What does this tell about the reachability matrix $\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right]$ and its rank. Show that there is a nonsingular matrix T such that

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] \quad \text{where} \quad A_{22} \in \mathbb{R}^{q\times q}.$$

Apply the above result to the following model:

$$A = \begin{bmatrix} -4 & 1 & -1 \\ -7 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}, \quad D = 0.$$

[2]. Given $A \in \mathbb{R}^{n\times n}$, show that the following three conditions are equivalent:

1. $A + A^* < 0$.
2. $\|e^{At}\|_2 < 1$, for $t > 0$.
3. $\exists\ \mathcal{Q} = \mathcal{Q}^* > 0$ such that $A\mathcal{Q} + \mathcal{Q}A^* < 0$ and $A^*\mathcal{Q} + \mathcal{Q}A < 0$.

Hint: Prove that $\|e^{At}\|_2^2 \leq e^{\lambda_{\max} t}$ for $t > 0$ where $\lambda_{\max}$ is the maximum eigenvalue of $(A + A^*)$. In proving this, you may consider the differential equation $\dot{X}(t) = AX(t)$ with some initial condition $X_0(t)$. Also, note that $\frac{d}{dt}(X^*X) = \dot{X}X + X^*\dot{X}$.

[3]. Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{k\times k}$ and $C \in \mathbb{R}^{n\times k}$, let $\mathcal{X}$ be the solution to the Sylvester Equation $A\mathcal{X} + \mathcal{X}B + C = 0$. Write your own routines to find $\mathcal{X}$ using **(i)** The Knocker product method, and **(ii)** The Schur-decomposition method. Apply your algorithms to compute $\mathcal{X}$ for the following two cases:

1. $A_1 = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 1 & 3 \end{bmatrix}.$
2. $A_2 = A_1$, $C_2 = C_1$, and $B_2 = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$

Do your algorithms work smoothly for both cases? If not, what is the reason for the breakdown?

[4]. Consider the following transfer function:

$$G(s) = \frac{-44.1s^3 + 334s^2 + 1034s + 390}{s^6 + 20s^5 + 155s^4 + 586s^3 + 1115s^2 + 1034s + 390}.$$

1. Using matlab (or equivalent) compute the Hankel singular values of $G(s)$. Moreover, compute a balancing realization for $G(s)$. Then obtain the reduced order models $G_k(s)$ of order $k = 2$ and $k = 4$ using balanced truncation.
2. Compute the $\mathcal{H}_\infty$ norms of the error models $G(s) - G_2(s)$ and $G(s) - G_4(s)$, and compare them with the theoretical upper bounds. Also, plot the amplitude Bode plots of $G(s)$, $G_2(s)$ and $G_4(s)$ (on top of each other for comparison). Similarly, show the amplitude Bode plots of of the error models $G(s) - G_2(s)$ and $G(s) - G_4(s)$ on a single figure.

[5]. Consider the system described by

$$H(s) = \sum_{i=1}^{n} \frac{a^{2i}}{s + a^{2i}}, \quad \text{where } a \in \mathbb{R} \text{ and } a > 0.$$

1. Compute a minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of $H(s)$ where A is diagonal and $B = C^T$. Based on this realization, show that the two infinite gramians $\mathcal{P}$ and $\mathcal{Q}$ can be chosen as

$$\mathcal{P}_{ij} = \mathcal{Q}_{ij} = \frac{a^{i+j}}{a^{2i} + a^{2j}}$$

where $\mathcal{P}_{ij}$ and $\mathcal{Q}_{ij}$ denote the (i,j) element of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Note that since $\mathcal{P} = \mathcal{Q}$, the realization is balanced (but not principal-axis balanced).
Before the 2^{nd} part of the question, we recall a well known result; for details see Section 5.5 of the lecture notes.
Lemma: Let σ_i, $i = 1, \ldots, n$ be the n distinct Hankel singular values of an asymptotically stable linear dynamical system $G(s)$. Then

$$\|G(s)\|_{\mathcal{H}_\infty} \leq 2(\sigma_1 + \sigma_2 + \cdots + \sigma_n).$$

2. Prove that the sum of the Hankel singular values of $H(s)$ is $\frac{n}{2}$. Moreover, show that the $\mathcal{H}_\infty$ norm of $H(s)$ is n. Hence in this case the upper bound in the above lemma is tight.
3. Take $n = 10$, $a = 1.25$ and reduce the order to $k = 4$ using balanced truncation. Compute the $\mathcal{H}_\infty$ norm of the error model and compare it with the theoretical upper bound. Also, plot the amplitude Bode plots of $H(s)$, the reduced model and the error model.

[6]. Consider a Chebychev-I continuous-time low-pass filter which can be obtained in Matlab as

$$[\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}] = \mathtt{cheby1}(\mathtt{n}, \mathtt{r}, 1, '\mathtt{s}'),$$

where r is the admisible ripple in the passband. For this question, take $r = 1$ db which corresponds approximately to 10% ripple. The first goal of this problem is to examine the numerical issues of the two different balancing schemes: Let T_1 denote the usual balancing transformation and T_2 denote the square-root balancing transformation. Vary the order of the filter as $n = 5 : 30$. For each n, compute the balancing transformations T_1 and T_2, and compute the following three errors

$$\left.\begin{array}{rcl} E_i^{(1)} & = & \|I_n - T_i T_i^{-1}\| \\ E_i^{(2)} & = & \|A_i \Sigma + \Sigma_i A_i^* + B_i B_i^*\| \\ E_i^{(3)} & = & \|A_i^* \Sigma + \Sigma_i A_i + C_i^* C_i\| \end{array}\right\} \text{ for } i = 1, 2 \qquad (1)$$

where A_i, B_i, and C_i are the state-space matrices of the system balanced by T_i and Σ is the balanced gramian, i.e. it is diagonal with Hankel singular values. Plot these errors as a function of n. Plot also the Hankel singular values for $n = 10,\ 20, 30$. Use logarithmic scale for y axis. Moreover compute the condition numbers of T_i for every n.

Now, for $n = 5 : 30$, reduce the order of the each filter to approximately $1/4$ of its original dimension, i.e. $k = round(n/4)$. However, the reduced models should be obtained directly without computing the full-order balanced model. Hence let $T_{ik} = [\ I_k \quad 0\ ]$ and $\hat{T}_{ik} = T_i^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix}$ for $i = 1,\ 2$. Note that, as we discussed in the class, T_{ik} and $\hat{T}_{ik}$ can be obtained in a more efficient way without forming T_i and T_i^{-1}. Obtain the reduced models using T_{ik} and $\hat{T}_{ik}$ and compute the above three errors for the reduced ordered quantities. Compare these errors with the previous ones. Also, compute the condition number of the each transformation. How do they compare with the condition numbers of the full-order transformations?

Finally, for $n = 30$, what are the possible dimensions of the reduced system obtained by balanced truncation to avoid the poles on the imaginary axis.

Note: To apply the square-root balancing transformation, you need the square-roots of the gramians. Provided "lyapSU.m" and "lyapSL.m" Matlab codes, thanks to Prof. D.C. Sorensen, precisely do this. However note that the both algorithms require the triple $(A,\ B,\ C)$ to be in the Schur-basis.

References

[1] K. Zhou with J. Doyle, *Essentials of Robust Control*, Prentice Hall, 1998.

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