ELEC 501 | | **Serkan Gugercin**
Approximation Theory | | October 4, 2002

Homework 2

Due Date: **October 25, 2002**

[1]. Let $G_1(s)$ and $G_2(s)$ be given two dynamical systems with state-space realizations $G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \in \mathbb{R}^{(n_1+p_1)\times(n_1+m_1)}$ and $G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \in \mathbb{R}^{(n_2+p_2)\times(n_2+m_2)}$.

a) Assuming compatible dimensions in each cases, find the state-space representations of $G_3(s) = G_1(s) + G_2(s)$ and $G_4(s) = G_1(s)G_2(s)$.

b) Now assume that $m_1 = p_1$ and D_1 is invertible. Prove that the state-space realization of the inverse of $G(s)$ is given by $G^{-1}(s) = \left[\begin{array}{c|c} A_1 - B_1D_1^{-1}C_1 & -B_1D_1^{-1} \\ \hline D_1^{-1}C_1 & D_1^{-1} \end{array}\right]$. Hint use the fact that $G^{-1}(s)G(s) = G(s)G^{-1}(s) = I_{m_1}$.

[2]. Consider the RLC network shown in Figure 1. The states variables are x_1, x_2, x_3 and x_4 as shown in the figure. The inputs are u_1 and u_2, and the outputs are $y_1 = x_2$ and $y_2 = x_4$.

a) Find the state-space representation of the system. Examine the reachability and observability of the system in terms of $R, R_1, R_2, C_1, C_2, L_1, L_2$. Is the system completely reachable and observable for all values of these parameters?

b) Take $R = R_2 = 1$, $R_1 = 2$, $L_1 = 1$, $L_2 = 6$, and $C_1 = C_2 = 1$. Is the system asymptotically stable? Is the system reachable using **(i)** only u_1, **(ii)** only u_2 and **(iii)** both inputs? If not, find a basis for the reachability subspace.

c) For the same numerical values as in part **b**, is the system observable by observing **(i)** only y_1, **(ii)** only y_2 and **(iii)** both outputs?. If not, find a basis for the unobservable subspace.

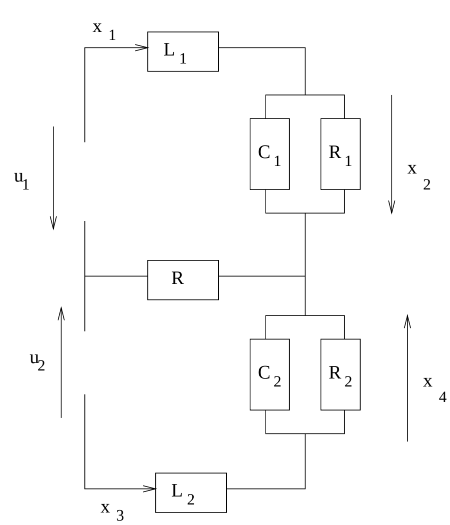

Figure 1: RLC Network

[3]. a) Consider the following dynamical system:

$$\mathbf{\Sigma} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 \\ \hline 1 & -3 & -5 & 4 & 0 \end{array}\right]$$

Assume that there is no input, i.e. $u(t) = 0$. Find *all* initial conditions $x(0)$ for which the states $x(t)$ will remain bounded as $t \to \infty$.

b) [1] A factor in determining useful life of a flexible structure, such as a ship, a tall building, or a large airplane, is the possibility of fatigue failures due to structural vibration. Each vibration mode is described by an equation of the form $m\ddot{x} + kx = u(t)$ where $u(t)$ is the input force and $m, k > 0$. Is it possible to find an input which will derive both the deflection x and the velocity $\dot{x}$ to zero in finite time for arbitrary initial conditions?

[4]. Let $\mathbf{\Sigma} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be an asymptotically stable, reachable and observable system. Let $\mathcal{P}$ and $\mathcal{Q}$ denote the infinite reachability an observability gramians of $\mathbf{\Sigma}$, i.e. the solutions to the following Lyapunov equations:

$$A\mathcal{P} + \mathcal{P}A^* + BB^* = 0 \quad \text{and} \quad A^*\mathcal{Q} + \mathcal{Q}A + C^*C = 0.$$

a) Show that in the frequency domain, $\mathcal{P}$ is given by

$$\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (jwI - A)^{-1}BB^*(jwI - A)^{-*}dw.$$

b) Let the discrete time system $\mathbf{\Sigma}_d = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ be obtained by applying bilinear transformation on $\mathbf{\Sigma}$. Let $\mathcal{P}_d$ and $\mathcal{Q}_d$ be, respectively, the infinite reachability and observability gramians of $\mathbf{\Sigma}_d$. Show that $\mathcal{P} = \mathcal{P}_d$ and $\mathcal{Q} = \mathcal{Q}_d$, i.e. the bilinear transformation preserves the infinite gramians.

c) A realization $\mathbf{\Sigma} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$ with $m = p$ is called symmetric if $D = D^*$ and if there exists a symmetric nonsingular matrix $S = S^*$ such that $AS = SA^*$ and $B = S^*C^*$. Now assume that $\mathbf{\Sigma}$ is asymptotically stable, reachable, observable and symmetric. The solution $\mathcal{X}$ to the Sylvester equation

$$A\mathcal{X} + \mathcal{X}A + BC = 0$$

is called *cross-gramian* of $\mathbf{\Sigma}$. Prove that $\mathcal{X}^2 = \mathcal{P}\mathcal{Q}$. Note that for $m = p = 1$, $\mathcal{X}^2 = \mathcal{P}\mathcal{Q}$ always holds.

[5]. Given $\mathbf{\Sigma} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$, the finite reachability gramian $\mathcal{P}(t)$ is defined as

$$\mathcal{P}(t) = \int_0^t e^{A\tau}BB^*e^{A^*\tau}d\tau.$$

Assume that $\mathbf{\Sigma}$ is asymptotically stable and the infinite reachability gramian $\mathcal{P}$ exists.

a) Prove that $\mathcal{P}(t) = \mathcal{P} - e^{At}\mathcal{P}e^{A^*t}$.

b) For a given time-interval $T = [\ t_1\ \ t_2\ ]$ where $0 \leq t_1 < t_2$, define the time-limited gramian $\mathcal{P}_T$ as

$$\mathcal{P}_T := \int_{t_1}^{t_2} e^{A\tau}BB^*e^{A^*\tau}d\tau.$$

Using part **a**, show that $\mathcal{P}_T$ satisfies a Lyapunov equation of the form $A\mathcal{P}_T + \mathcal{P}_T A^* + W = 0$. What is W?

[6]. The following differential equations describes the dynamics of an hot-air balloon:

$$\begin{aligned} \dot{\theta} &= -\tfrac{1}{\tau_1}\theta + u \\ \dot{v} &= -\tfrac{1}{\tau_2}v + \sigma\theta + \tfrac{1}{\tau_2}w \\ \dot{h} &= v \end{aligned}$$

θ : temperature change from equilibrium
v : vertical velocity
u : proportional to heat added to air in balloon (control input)
w : vertical wind velocity (distrubance input)
h : change in altitude from equilibrium

where $\tau_1 = 2$, $\tau_2 = 1$ and $\sigma = 2$.

a) Choosing θ, v and h as the state-variables, u and w as the inputs and $y = h$ as the output, find the state-space realization.

a) Let $w = 0$, i.e. there is only one input. Find the minimal energy inputs which transfer states from $x_1 = [\ 0\ \ 1\ \ 1\ ]^T$ to $x_2 = [\ 0\ \ 0\ \ 2\ ]^T$, i.e. to double the height and bring the speed to zero, in $T = 1$ unit of time and $T = 10$ unit of time? Plot the corresponding inputs and find the corresponding energies?

[7]. Consider the system given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) Find the transfer function of this system. Find all initial conditions such that for $u_1(t) = 0$ and $u_2(t) = e^{-2t}$, $t \geq 0$, the second output $y_2(t)$ is a constant multiple of the input $u_2(t)$. Compute this constant.

(b) Determine the controllability from the following inputs: (i) u_1, (ii)u_2, and (iii) u. If any of these systems is not controllable, find a basis for the controllable space. Compute the infinite controllability gramians and hence deduce the energy required to reach the i^{th} unit vector e_i (if e_i is in the reachability space) using u_1, u_2 and u, respectively. Comment on your results.

(c) Determine the observability from each of the outputs: (i) y_1, (ii) y_2, and (iii) y. If these systems are not observable determine the unobservable space. What can be said about the observable space? Compute the infinite observability gramians and hence deduce the energy obtained by observing the i^{th} unit vector e_i using y_1, y_2 and y. Comment on your results.

[8]. a) [2] Let $G(s) \in \mathcal{H}_\infty$. Prove that

$$\left\| \begin{bmatrix} G \\ I \end{bmatrix} \right\|_{\mathcal{H}_\infty}^2 = \|G\|_{\mathcal{H}_\infty}^2 + 1.$$

b) Compute the Hankel singular values, $\mathcal{H}_\infty$ norm, $\mathcal{H}_2$ norm and Hankel norm of the following systems:

$$G_1(s) = \left[\begin{array}{cc|c} -1 & 0 & 1 \\ 2 & -3 & -1 \\ \hline 1 & 2 & 0 \end{array}\right] \quad \text{and} \quad G_2(s) = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -2 & -3 & -4 & 1 \\ \hline 1 & 4 & 5 & -1 & 0 \end{array}\right]$$

References

[1] W.L. Brogan, *Modern Control Theory*, Prentice Hall, 3^{rd} Edition, 1991.

[2] K. Zhou with J. Doyle, *Essentials of Robust Control*, Prentice Hall, 1998.

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