**ELEC 501** **S. Gugercin**

Approximation Theory

# Homework 1

Due Date: **September 27, 2002**

**Problem [1].** Find the singular value decompositions and compute the rank one approximations, which are optimal in the 2-norm, of the matrices liste d below. Use paper and pencil.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix}$$

**Solution**

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & -1\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T$$

Rank one approximations $\hat{A}_i$, $i = 1, 2$ of the matrices:

$$\hat{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \mathbf{0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\hat{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T = \begin{pmatrix} 6 & 3 \\ 0 & 0 \end{pmatrix}$$

**Problem [2].** **(a)** Let $A \in \mathbb{R}^{n\times n}$ and $\det A \neq 0$. What is the relationship between the singular values of $A$ and $A^{-1}$?

**(b)** Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A$. Show that $\sigma_n(A) \leq |\lambda| \leq \sigma_1(A)$.

**(c)** Find the singular values of $A = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$. Explain your answer geometrically. Find $u \in \mathbb{R}^2$, $||u||_2 = 1$, such that $||Au||_2 = \sigma_1$. Explain.

**Solution:** **(a)** Let the SVD of the matrix $A$ be $A = USV^T$, where $U$ and $V$ are two orthogonal matrices, and $S$ is a diagonal matrix whose diagonal elements are the singular values of $A$. Since $A$ is nonsingular, $S$ must be nonsingular, and therefore, we have $A^{-1} = VS^{-1}U^T$. This means that the diagonal elements of $S^{-1}$, i.e., the reciprocals of the singular values of $A$, are the singular values of $A^{-1}$.

**(b)** Let $x$ be an eigenvector of the matrix $A$ corresponding to the eigenvalue $\lambda$. Clearly, we have $Ax = \lambda x$. Therefore,

$$x^T A^T A x = (Ax)^T (AX) = (\lambda x)^T (\lambda x) = \lambda^2 x^T x.$$

It follows from the facts

$$\sigma_{\max}(A) = \sqrt{\max_{x\neq 0} \frac{x^T A^T A x}{x^T x}} \qquad \sigma_{\min}(A) = \sqrt{\min_{x\neq 0} \frac{x^T A^T A x}{x^T x}}$$

that

$$\sigma_{\max}(A) \geq \sqrt{\frac{x^T A^T A x}{x^T x}} = |\lambda| \geq \sigma_{\min}(A)$$

holds true.

**(c)** The singular values of $A$ are square root the eigenvalues of $AA^T$:

$$AA^T = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \begin{pmatrix} p & -q \\ q & p \end{pmatrix}^T = \begin{pmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{pmatrix} = (p^2+q^2)\, I_2.$$

Therefore, the two singular values of $A$ are both equal to $\sqrt{p^2+q^2}$.
The vector $u \in \mathbb{R}^2$, $||u||_2 = 1$ satisfying the condition $||Au||_2 = \sigma_1$ is the right singular vector of $A$ corresponding to the largest singular value of $A$. Since the two singular values of $A$ are equal, the two right singular vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are vectors satisfying the conditions.

**Problem [3]. (a)** Show by direct computation that the **LS** solution of $Ax = b$ is given by

$$x_{LS} = (A^T A)^{-1} A^T b, \qquad A \in \mathbb{R}^{n\times m},\ n \geq m,\ \text{rank}A = m$$

**(b)** Using the **SVD** of $A$ show that

$$x_{LS} = \sum_{i=1}^{m} \frac{u_i^* b}{\sigma_i} v_i, \qquad \text{and} \qquad ||Ax_{LS} - b||^2 = \sum_{i=m+1}^{n} (u_i^* b)^2.$$

What is the geometrical interpretation of this problem? Do $Ax_{LS}$ and $b - Ax_{LS}$ lie in the Im$(A)$, Ker$(A)$ or in a space perpendicular to Im$(A)$? Justify your results.

**Solution: (a)** The problem is formulated as an optimization problem:

$$J(x) = ||Ax - b||^2 = (Ax-b)^T(Ax-b).$$

Compute the partial derivatives:

$$\begin{aligned} \frac{\partial J}{\partial x_i} &= (Ae_i)^T(Ax-b) + (Ax-b)^T(Ae_i) \\ &= 2e_i^T A^T(Ax-b) = 2e_i^T(A^TAx - A^Tb), \quad i = 1,2,\ \ldots,\ m. \end{aligned}$$

Therefore, the optimizer $x$ satisfies the condition

$$\frac{\partial J}{\partial x_i} = 2e_i^T(A^TAx - A^Tb) = 0, \quad i=1,2,\ \ldots,\ m \iff A^TAx - A^Tb = 0$$

$$\iff x = (A^TA)^{-1}A^Tb$$

Since the Hessian matrix of $J(x)$ is $A^TA$, which is nonnegative definite, the optimizer is the LS solution of $Ax = b$.

**(b)** Let $A$ have the following SVD:

$$A \;=\; U\Sigma V^T = [\, U_1 \;\; U_2 \,] \begin{bmatrix} \Sigma_1 \\ \mathbf{0} \end{bmatrix} V^T = U_1\Sigma_1 V^T \qquad \text{where} \qquad U_1 \in \mathbb{R}^{m\times n},\; \Sigma_1 \in \mathbb{R}^{n\times n}$$

Then it follows that

$$\begin{aligned} J(x) &= \|Ax-b\|^2 = \|U_1\Sigma_1V^Tx - b\|^2 = \|U_1\Sigma_1V^Tx - U_1U_1^Tb + (I-U_1U_1^T)b\|^2 \\ &= \|\underbrace{U_1(\Sigma_1V^Tx - U_1^Tb)}_{y} + \underbrace{(I-U_1U_1^T)b}_{z}\|^2 \end{aligned}$$

Note that $y^Tz = 0$. Hence $J(x)$ can be written as

$$J(x) \;=\; \underbrace{\|U_1(\Sigma_1V^Tx - U_1^Tb)\|^2}_{J_1(x)} + \underbrace{\|(I-U_1U_1^T)b\|^2}_{J_2(x)} =$$

One should notice that $J_2(x)$ is independent of $x$. Hence $J(x) \ge J_2(x)$ for $\forall x$. On the other hand, $J_1(x)$ can be made zero by choosing $x = x_{LS} = V\Sigma_1^{-1}U^Tb$. Therefore the minimum of $J(x)$ is obtained for $x = x_{LS}$ and equal to

$$\min J(x) = J(x_{LS}) = \|Ax_{LS} - b\|^2 = J_2(x) = \|(I-U_1U_1^T)b\|^2$$

It is straightforward to show that $x_{LS} = V\Sigma_1^{-1}U^Tb = \sum_{i=1}^m \frac{u_i^*b}{\sigma_i}v_i$. Moreover, the observation $U_1U_1^T + U_2U_2^T = I$ simply leads to the second equality

$$\|Ax_{LS} - b\|^2 = \sum_{i=m+1}^{n} (u_i^*b)^2.$$

These results can also be obtained by directly computing $x_{LS} = (A^TA)^{-1}A^Tb$.

To answer the second part of the problem, we compute

$$\begin{aligned} Ax_{LS} &= U_1\Sigma_1V^TV\Sigma_1^{-1}U^Tb \\ &= U_1U_1^Tb \end{aligned} \qquad \begin{aligned} b - Ax_{LS} &= b - U_1U_1^Tb \\ &= (I-U_1U_1^T)b \end{aligned}$$

Hence $Ax_{LS}$ lie in the Im$A$ and $b - Ax_{LS}$ lie in a space perpendicular to Im$(A)$. Geometrically, we decompose $b$ into two parts:

$$b = \underbrace{U_1U_1^Tb}_{b_1} + \underbrace{(I-U_1U_1^T)b}_{b_2}$$

where $b_1$ lie in the Im$(A)$ and $b_2$ is perpendicular to the Im$(A)$. Then the least square solution solution $x_{LS}$ is obtained by solving the linear system $Ax_{LS} = b_1$ and the least square residual is equal to $\|b_2\|$. Note that when $m = n$, $b_2 = 0$.

**Problem [4]. (a)** Prove that if $A \in \mathbb{R}^{m\times n}$, then there exists a unit 2-norm vector $z \in \mathbb{R}^n$ such that $A^TAz = \sigma_1^2 z$ where $\sigma_1 = \|A\|_2$.

**(b)** Using the above result show that if $A \in \mathbb{R}^{m\times n}$, then $\|A\|_2 \le \sqrt{\|A\|_1\|A\|_\infty}$.

**(c)** Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Show that that if $E = uv^T$ and $v^T u = 1$, then $\|E\|_F = \|E\|_2 = \|u\|_2 \|v\|_2$.

**Solution:** **(a)**Let $A$ have singular value decomposition $U\Sigma V^T$, with the first column of $V$ given by $v_1$. Then

$$A^T A v_1 = V\Sigma U^T U \Sigma V^T v_1 = V\Sigma^2 e_1 = \sigma_1^2 v_1$$

Thus we can take $z = v_1$ to get the desired result.
**(b)** Let $v \neq 0$ be such that $A^T Av = \sigma_1^2 v$ where $\sigma_1 = \|A\|_2$. Then we obtain

$$\|A^T Av\|_1 \leq \|A^T\|_1 \|A\|_1 \|z\|_1 = \|A\|_\infty \|A\|_1 \|z\|_1.$$

Then the desired result simply follows.

**(c)** First observe that $E = uv^T$ has rank 1. Hence, $\sigma_i = 0$ for $i = 2, \ldots, n$. This observation immediately yields that $\|E\|_F = \|E\|_2 = \sigma_1(E)$. Moreover we have

$$\begin{aligned} \|E\|_F^2 &= trace(E^T E) = trace(vu^T uv^T) = u^T u \, trace(y^T y) \\ &= \|u\|_2^2 \, \|v\|_2^2 \end{aligned}$$

The last equality simply leads to $\|E\|_F = \|E\|_2 = \|u\|_2 \|v\|_2$.

**Problem [5].** **(a)** For any arbitrary matrix $A$, let $E$ be a matrix such that $\|E\|_2 < \sigma_{min}(A))$. Then prove that $rank(A+E) \geq rank(A)$.

**(b)** Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if rank($A$)=$n$, then $\|x\|_A = \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

**Solution:** **(a)**Suppose $rank(A) = k$ and $rank(A+E) = m$ , now if $m < k$ then $A + E$ is a low-rank approximation of $A$ and because of Schmidt-Mirsky theorem we have:

$$\|E\|_2 = \|A - (A+E)\|_2 \geq \sigma_{min}(A)$$

which is a contradiction.

**(b)** We need to show that the norm $\|x\|_A = \|Ax\|$ satisfies strict positiveness, triangular inequality, and positive homogeneity.

- **strict positiveness:** Let $x$ be any vector in $\mathbb{R}^n$. Then $\|x\|_A = \|Ax\| \geq 0 \; \forall x$ since $\|\cdot\|$ is a vector norm in $\mathbb{R}^m$. But we also need to show that $\|x\|_A = 0$ holds iff $x = 0$. Let $x = 0$. Then it follows that $\|x\|_A = 0$ as required. Now assume $\|x\|_A = \|Ax\| = 0$. Since $\|\cdot\|$ is vector norm, this yields that $Ax = 0$. But we know that $A$ has full rank. Hence $Ax = 0$ iff $x = 0$. So, strict positiveness is satisfied.
- **Triangular inequality** $\|x+y\|_A = \|Ax + Ay\|$. $\|\cdot\|$ is a vector norm, so it follows that $\|x+y\|_A \leq \|Ax\| + \|Ay\| = \|x\|_A + \|y\|_A$ as required by the triangular inequality.
- **positive homogeneity:** It is straightforward to show that $\|\alpha x\|_A = |\alpha| \, \|x\|_A$

Hence, $\|x\|_A$ is a vector norm in $\mathbb{R}^n$.

**Problem [6].** **(a)** Show that the Frobenius norm, i.e. the Schatten 2-norm,

$$\|A\|_F = \left(\sum_{i=1}^{n} \sigma_i^2(A)\right)^{\frac{1}{2}} \quad \text{where} \quad A \in \mathbb{R}^{n\times m}, \quad n \leq m$$

is unitary invariant, that is $\|A\|_F = \|UAV\|_F$ where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ with $UU^T = I_n$ and $VV^T = I_m$. Also, prove that

$$\|A\|_F^2 = \text{trace}\,(A^T A) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mid A(i,j) \mid^2$$

**(b)** Show that the Shatten 1-norm, also known as the trace norm,

$$|||A|||_1 = ||A||_{\text{trace}} := \sum_{i=1}^{n} \sigma_i(A), \quad \text{where} A \in \mathbb{R}^{n\times m}, \quad n \le m$$

satisfies the triangle inequality.
**Hint:** Use the following Lemma:
**Lemma:** Let $A \in \mathbb{R}^{n\times n}$ have singular values $\sigma_i(1)$, $i = 1,\ldots,n$ in decreasing order and $C \in \mathbb{R}^{n\times n}$ be a rank $k$ partial isometry. Then for each $k = 1,\ldots,n$, we have

$$\sum_{i=1}^{k} \sigma_i(A) = \max\{\mid trace(AC) \mid\}$$

**Solution:** **(a)** Let $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ be such that $UU^T = I_n$ and $VV^T = I_m$. Let $A$ have singular value decomposition given by $A = \bar{U}S\bar{V}^T$. Note that $(U\bar{U})(U\bar{U})^T = I$, similarly for $V, \bar{V}$. Thus the matrix $UAV^T$ has the same singular values as $A$, and so the same Frobenius norms.

The first equality follows from the definition of frobenius norm and unitary invariance of said norm; the second equality from the definition of frobenius norm.
**(b)** Zero-fill A,B to make them both $m \times m$. Let $A + B$ have svd $U\Sigma V$ with $U$ rank n partial isometry, $V$ rank m partial isometry.

$$\begin{aligned}
|||A+B||| &= tr(U^*(A+B)V) \\
&= tr(U^*AV) + tr(U^*BV) \\
&\le \sum_{i=1}^{m} \sigma_i(U^*A) + \sum_{i=1}^{m} \sigma_i(U^*B) \\
&\le \sum_{i=1}^{m} \sigma_i(U^*)\sigma_i(A) + \sum_{i=1}^{m} \sigma_i(U^*)\sigma_i(B) \\
&= \sum_{i=1}^{n} \sigma_i(A) + \sum_{i=1}^{n} \sigma_i(B) \\
&= |||A||| + |||B|||
\end{aligned}$$

**Problem [7].** Approximation of the **clown.mat** image using **Matlab**: after starting **Matlab**, type

```
load clown;
Z = ind2gray(X, map);
[mz,nz] = size(Z);
imshow(Z,64);
mag = 2;
truesize(1, [mz*mag, nz*mag]);
```

Compute the **SVD** of $Z$, using **svd**: `[U,S,V] = svd(Z);`

1. Plot the singular values of $Z$ on a logarithmic scale.
2. Compute approximant having error less than 10%, 5%, 2% of the largest singular value of $Z$. What is the rank of the corresponding approximants? also for each case compute the compression ratio (compression ratio is defined as the number of bytes required to store the approximant devided by the original image size in bytes.)
3. Now tile the image into four equal pieces. for each of the above errors, use SVD to approximate the sub-images and then reconstruct the complete image from them. Compute the compression ratio for this image. Compute the 2-norm error of the approximant and then compare the result with the previous one. Which method is better? which one requires more computations?
4. Attach with your homework the original image, the approximant from the first method and second method, for the case the error is less than 2%.

**Solution:** please refer to the following code for the comparison between the two methods:

```
clear
% Loading the image ...
load clown;
Z=ind2gray(X,map);
[mz,nz]=size(Z);
% Showing the original image
figure(1)
imshow(Z,64);

% Comuting the SVD and the truncation index k

[U,S,V]=svd(Z);
s=diag(S);

figure(2)
semilogy(s/s(1),'.-');
xlabel('Number');
ylabel('Singualr Value');
grid;

% Computing the rank of the approximant (truncation index)

percent=0.02;
k=sum(s/s(1)>=percent);
S1=zeros(size(S));
S1(1:k,1:k)=S(1:k,1:k);
Z1=U*S1*V';

% For 10% error, rank of the approximant is 7.
% For  5% error, rank of the approximant is 14.
% For  2% error, rank of the approximant is 40.



% Show the compressed image using the regular method for 2% error

figure(3)
imshow(Z1,64)
```

```
% Tiling the image
%
% Z= [Z11,Z12 ; Z21,Z22];

Z11=Z(1:mz/2,1:nz/2);1
Z12=Z(1:mz/2,nz/2+1:nz);
Z21=Z(mz/2+1:mz,1:nz/2);
Z22=Z(mz/2+1:mz,nz/2+1:nz);

% truncating each part

[U11,S11,V11]=svd(Z11);
s11=diag(S11);
k11=sum(s11/s11(1)>=percent);
S11_1=zeros(size(S11));
S11_1(1:k11,1:k11)=S11(1:k11,1:k11);
Z11_1=U11*S11_1*V11';

[U12,S12,V12]=svd(Z12);
s12=diag(S12);
k12=sum(s12/s12(1)>=percent);
S12_1=zeros(size(S12));
S12_1(1:k12,1:k12)=S12(1:k12,1:k12);
Z12_1=U12*S12_1*V12';

[U21,S21,V21]=svd(Z21);
s21=diag(S21);
k21=sum(s21/s21(1)>=percent);
S21_1=zeros(size(S21));
S21_1(1:k21,1:k21)=S21(1:k21,1:k21);
Z21_1=U21*S21_1*V21';

[U22,S22,V22]=svd(Z22);
s22=diag(S22);
k22=sum(s22/s22(1)>=percent);
S22_1=zeros(size(S22));
S22_1(1:k22,1:k22)=S22(1:k22,1:k22);
Z22_1=U22*S22_1*V22';

% reconstruncting the tiled image

ZZ=zeros(size(Z));
ZZ(1:mz/2,1:nz/2)=Z11_1;
ZZ(1:mz/2,nz/2+1:nz)=Z12_1;
ZZ(mz/2+1:mz,1:nz/2)=Z21_1;
ZZ(mz/2+1:mz,nz/2+1:nz)=Z22_1;

% Showing the compressed image using Tiling method
figure(4)
imshow(ZZ,64)
```

```
% Computing Compression ratio for Regular Method
CR =(mz+nz+1)*k/(2*mz*nz+min(mz,nz));
% Computing 2-norm error of the compressed image
error=norm(Z-Z1);

% Computing Compression ratio for Tiling Method
K=[k11,k12,k21,k22];
% Computing 2-norm error of the compressed image
CR_1=sum((mz/2+nz/2+1)*K/(2*mz*nz+4*min(mz/2,nz/2)))
error1=norm(Z-ZZ);

end

% normalizing the errors

error=error/norm(Z);
error1=error1/norm(Z);
```

To compute the compession ration, one can use this formula:

$$CR = \frac{(m+n+1))k}{(2mn + min(m,n))}$$

where $k$ is the approximants rank and $m$ and $n$ are the size of the image. Figure 5 shows the comparison between the two method and it can be seen from this plot that the tiling method is not an effective way to compress. It also turns out that numerically it is more complex than the regular method.

**Problem** [8]. Denoising and sorting of images using the **SVD**:
Given are the images of of the three numbers **1**, **7** and **0** represented in Figure-1 (12-pixel images):

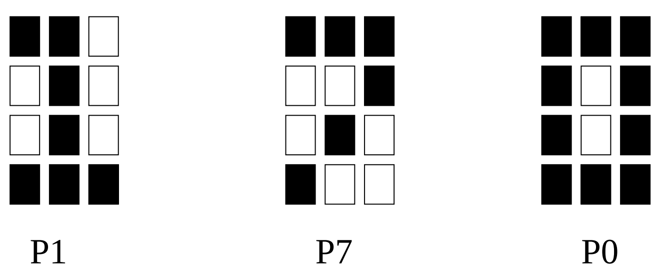


Figure 1: Numbers '1', '7' and '0'

1. Convert the three images in Figure-1 to three $4 \times 3$ matrices by replacing black pixels by 1 and white pixels by 0. Let the matrices be called $P1$, $P7$ and $P0$.
2. **Obtaining the noisy data:** Construct 500 noisy versions of each image as follows:

   `P1+0.05*rand(4,3); P7+0.05*rand(4,3); P0+0.05*rand(4,3);`

   This yields 1500 noisy data matrices $N_i \in \mathbb{R}^{4\times3}$, $i = 1, \cdots, 1500$.

3. Vectorize each image $N_i$ as follows:

$$x_i := [\ N_i(:,1);\ \ N_i(:,2);\ \ N_i(:,3)\ ] \in \mathbb{R}^{12\times 1}, \qquad i = 1, \cdots, 1500.$$

Then, form the following matrix: $A = [\ x_1\ \ x_2\ \ x_3\ \ \cdots\ \ x_{1500}] \in \mathbb{R}^{12\times 1500}$.

4. Take the SVD of A and examine the singular values. How many significant singular values do you see? Find the best rank 3 approximation of A in the 2-norm. Let $A_3$ be the optimal approximant with SVD $A_3 = U_3\Sigma_3 V_3^T$ where $U_3 \in \mathbb{R}^{12\times 3}$, $\Sigma_3 \in \mathbb{R}^{3\times 3}$ and $V_3 \in \mathbb{R}^{1500\times 3}$. Each column of $V_3^T$ corresponds to a point in 3-dimensional space. Plot all 1500 points each of which corresponds to a column of $V_k^T$. You should obtain 3 clusters of points. Find the center of gravity of these 3 clusters. Now, using the coordinates of these points form the $3\times 3$ matrix $W_3^T$. And finally compute the matrix $\hat{A} = U_3\Sigma_3 W_3^T \in \mathbb{R}^{12\times 3}$.

5. Reverse the procedure in Step-2 and obtain the matrices $\hat{P}1$, $\hat{P}7$, and $\hat{P}0 \in \mathbb{R}^{4\times 3}$. Compare these matrices with the original noise-free data matrices $P1$, $P7$ and $P0$. Comment on your results.

**Solution:**

The following is working matlab code for this problem.

```
%The data:
P1=[1 1 0;0 1 0;0 1 0;1 1 1];
P7=[1 1 1;0 0 1;0 1 0;1 0 0];
P0=[1 1 1;1 0 1;1 0 1;1 1 1];

%2.& 3.
An=zeros(12,1500);
for i=1:500
P1n=P1+.05*rand(4,3);
An(:,i*3-2)=reshape(P1n,12,1);

P7n=P7+.05*rand(4,3);
An(:,i*3-1)=reshape(P7n,12,1);

P0n=P0+.05*rand(4,3);
An(:,i*3)=reshape(P0n,12,1);
end

%4.
[U,S,V]=svd(An);
%Look at the singular values. There should be 3 singular values substantially
%larger than the others.

A3=U(:,1:3)*S(1:3,1:3)*V(:,1:3)';
%To plot the columns of V, use the command:
%plot3(V(:,1),V(:,2),V(:,3),'.')
%If we knew nothing about the data we would set up partitions and at each
%step test to see where the point in question belongs. However, we know
%(expect) that every our points will alternate between the cogs.

V3=V(:,1:3);
COG=zeros(3,3);
for i=0:499
  COG(1,:)=V3(1+i*3,:)+COG(1,:);
  COG(2,:)=V3(2+i*3,:)+COG(2,:);
```

```
  COG(3,:)=V3(3+i*3,:)+COG(3,:);
end
COG=COG/500;
%hold
%plot3(COG(:,1),COG(:,2),COG(:,3),'r+');
%figure;
%plot3(COG(:,1),COG(:,2),COG(:,3),'r+');

A_hat=U(:,1:3)*S(1:3,1:3)*COG';
P1_hat=reshape(A_hat(:,1),4,3);
P7_hat=reshape(A_hat(:,2),4,3);
P0_hat=reshape(A_hat(:,3),4,3);
```