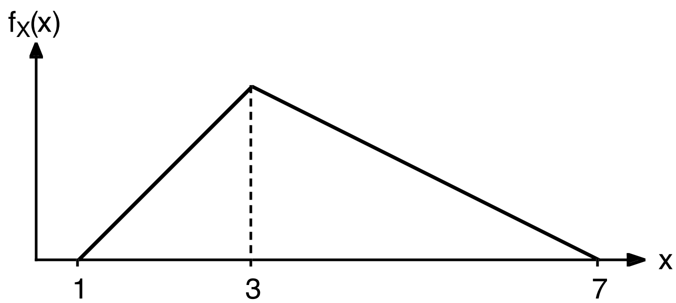
First Exam Solutions

1. Describe an algorithm (in sufficient detail that someone unfamiliar with random number generation could implement it as a program) for generating random numbers for a random variable X with a density function $f_X(x)$ shown in the figure below. You may assume that the person who will implement the generator has access to and knows how to use a uniform random number generator. There are several ways to do this; choose an efficient method. (25 pts)



You can generate random numbers for this density function with several methods. Probably the most straightforward is to use the acceptance/rejection method with a constant bounding function. Since the area under the curve must be one (it is a density function, after all), the peak of the density function is $f_X(3) = 1/3$. The density function can be represented as

$$f_X(x) = \begin{cases} \frac{1}{6}(x-1) & 1 \le x \le 3 \\ \frac{1}{12}(7-x) & 3 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

One approach using acceptance/rejection would be to transform X into a random variable Y that is defined on $(0,1)$ and use a $U(0,1)$ density function scaled by 3 to bound it. Let

$$Y = \frac{X-1}{6}$$

Then

$$f_Y(y) = \begin{cases} y & 0 \le y \le \frac{1}{3} \\ \frac{1}{2}(1-y) & \frac{1}{3} \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Generate two $U(0,1)$ random variables Y_1 and Y_2. If $Y_2 \le 3f_Y(Y_1)$, then accept $Y = Y_1$; otherwise, repeat. Once a value for Y is accepted, transform it back to X. The downside of this approach is that it requires two iterations on the average to generate one acceptable random number, and each iteration must generate two $U(0,1)$ random numbers.

We can improve the efficiency of the acceptance/rejection method by using a better bounding function. The only requirement for the bounding function (other than being a better bound than a constant) is that it be easy to generate a random variable with a distribution that is a scaled version of the bounding function. Usually, this means that the bounding function should be invertible. Two functions should immediately suggest themselves – $g_1(x) = \frac{1}{6}(x-1)$ and $g_2(x) = \frac{1}{12}(7-x)$. It should be obvious from a sketch of these functions and the original density function that $g_2(x)$ will be better because the area under $g_2(x)$ in the region $(1,7)$ will be smaller than the area under $g_1(x)$. Without going into detail (see the text), the average number of iterations per accepted value will be 1.5 instead of two. However, each iteration still requires two $U(0,1)$ random variables.

A better approach is to generate a value for X directly, either using straight inversion or by treating the density function for X as the weighted sum of two simpler density functions (composition) combined with inversion. Since the methods are quite similar, only straight inversion will be demonstrated.

$$F_X(x) = \begin{cases} \dfrac{(x-1)^2}{12} & 1 \le x \le 3 \\ 1 - \dfrac{(7-x)^2}{24} & 3 < x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

Apply inversion to each region separately. Generate a value z for a $U(0,1)$ random variable. If $0 \le z \le 1/3$, use $x = \sqrt{12z} + 1$. Otherwise, use $x = 7 - \sqrt{24(1-z)}$. Note that you cannot use z here in place of $1-z$. This is more efficient than the acceptance/rejection method, since it requires only one $U(0,1)$ random number for each value of X.

2. This problem refers to the model of a shared-memory multiprocessor presented in class. Use the information for that model to develop a discrete-time Markov chain implementation of the model for a system with five processors and three memory modules. Specify the Markov chain graphically, being sure to label (clearly!) all states and include the transition probabilities on the arcs representing the transitions between states. Show your work on deriving the state transition probabilities. Do not solve for the steady state probabilities. (25 pts)

The Markov chain requires five states: (500), (410), (320), (311), and (221). The state transitions and associated probabilities are:

500 ⇒ 500: $\frac{1}{3}\cdot\binom{1}{1} = \frac{1}{3}$

500 ⇒ 500: $\frac{1}{3}\cdot\binom{2}{1} = \frac{2}{3}$

410 ⇒ 500: $\left(\frac{1}{3}\right)^2\cdot\binom{1}{1} = \frac{1}{9}$

410 ⇒ 410: $\left(\frac{1}{3}\right)^2\cdot\binom{2}{1}\cdot 2! = \frac{4}{9}$

410 ⇒ 320: $\left(\frac{1}{3}\right)^2\cdot\binom{2}{1} = \frac{2}{9}$

410 ⇒ 311: $\left(\frac{1}{3}\right)^2\cdot\binom{2}{2}\cdot 2! = \frac{2}{9}$

320 ⇒ 410: $\left(\frac{1}{3}\right)^2\cdot\binom{1}{1} = \frac{1}{9}$

320 ⇒ 320: $\left(\frac{1}{3}\right)^2\cdot[2!+1] = \frac{3}{9}$

320 ⇒ 311: $\left(\frac{1}{3}\right)^2\cdot 2! = \frac{2}{9}$

320 ⇒ 221: $\left(\frac{1}{3}\right)^2\cdot[2!+1] = \frac{3}{9}$

311 ⇒ 500: $\left(\frac{1}{3}\right)^3\cdot\binom{1}{1} = \frac{1}{27}$

311 ⇒ 410: $\left(\frac{1}{3}\right)^3\cdot\binom{2}{1}\cdot\binom{3}{2} = \frac{6}{27}$

311 ⇒ 320: $\left(\frac{1}{3}\right)^3\cdot\left[\binom{2}{1}+\binom{2}{1}\cdot\binom{3}{2}\right] = \frac{8}{27}$

311 ⇒ 311: $\left(\frac{1}{3}\right)^3\cdot 3! = \frac{6}{27}$

311 ⇒ 221: $\left(\frac{1}{3}\right)^3\cdot\binom{3}{2}\cdot 2! = \frac{6}{27}$

221 ⇒ 410: $\left(\frac{1}{3}\right)^3\cdot\binom{2}{1} = \frac{2}{27}$

221 ⇒ 320: $\left(\frac{1}{3}\right)^3\cdot\binom{3}{2}\cdot 2! = \frac{6}{27}$

$221 \Rightarrow 311$: $\left(\frac{1}{3}\right)^3 \cdot \left[\binom{1}{1} + \binom{2}{1} \cdot \binom{3}{2}\right] = \frac{7}{27}$

$221 \Rightarrow 221$: $\left(\frac{1}{3}\right)^3 \cdot \left[\binom{3}{3} \cdot 3! + \binom{2}{1} \cdot \binom{3}{1}\right] = \frac{12}{27}$

3. (a) Solve the equation $\underline{\pi}\underline{\underline{P}} = \underline{\pi}$ for the discrete-time Markov chain described by the following single-step transition probability matrix, assuming that the elements of $\underline{\pi}$ are all non-negative and sum to 1. (You need to be a little bit careful here – fools rush in where matlab fears to tread.) (10 pts)

$$\underline{\underline{P}} = \begin{bmatrix} 0 & 0.2 & 0 & 0.8 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.8 & 0 & 0.2 & 0 \end{bmatrix}$$

The Chapman-Kolmogorov equations are:

$$\pi_1 = 0.6\pi_1 + 0.8\pi_4$$
$$\pi_2 = 0.2\pi_1 + 0.4\pi_3$$
$$\pi_3 = 0.4\pi_2 + 0.2\pi_4$$
$$\pi_4 = 0.8\pi_1 + 0.6\pi_3$$

Adding the first and third (or the second and fourth) C-K equations:

$$\pi_1 + \pi_3 = \pi_2 + \pi_4$$

Since the elements of $\underline{\pi}$ sum to 1,

$$\pi_1 + \pi_3 = \pi_2 + \pi_4 = 0.5$$

From this, we get

$$\pi_1 = 0.5 - \pi_3$$
$$\pi_2 = 0.5 - \pi 4$$

Substituting for π_1 and π_2 in the first two C-K equations leads to

$$\pi_3 = 0.2 - 0.2\pi_4$$
$$\pi_3 = 2 - 5\pi_4$$

and hence

$\underline{\pi} = \begin{pmatrix} 3/8 & 1/8 & 1/8 & 3/8 \end{pmatrix}$

(b) Why is $\underline{\pi}$ **not** the steady-state probability vector for the Markov chain? (5 pts)

Because the Markov chain is periodic. Draw the state diagram, or note that $\underline{\underline{P}}^{2i}$ and $\underline{\underline{P}}^{2i+1}$ both have "checkerboard" patterns of zero and non-zero elements, such that the patterns always alternate.

(c) What does $\underline{\pi}$ represent in this case? To put it another way, when a unique solution $\underline{x}$ to the equation $\underline{x}\underline{\underline{P}} = \underline{x}$ exists, what is a meaningful interpretation of $\underline{x}$ if it isn't the steady-state probability vector? Be specific. Explain what the equation $\underline{x}\underline{\underline{P}} = \underline{x}$ means in this case. (10 pts)

Suppose we observed n transitions of the Markov chain, and counted the number of transitions n_i out of state i, $i = 1, 2, 3$, and 4. Then $x_i = \lim_{n\to\infty} n_i/n$, if this limit exists, is the fraction of transitions out of state i in the long run. For every transition out of state i, there must be a transition into state i. The fraction of transitions from state j into state i, given that the state Is j, is just p_{ji}. Hence,

$$x_i = p_{1i}x_1 + p_{2i}x_2 + p_{3i}x_3 + p_{4i}x_4,\ i = 1, 2, 3, \text{ and } 4$$

or $\underline{x}\underline{\underline{P}} = \underline{x}$. If we assume that each transition takes one time unit, x_i is also the fraction of time the chain spends in state i.

4. Outtel is a manufacturer of microprocessors. Its Septium 2000 line of microprocessors has a tiny market share, mainly because the Septium 2000 chips have the reputation of being incredibly slow. In an effort to counter the word-of-mouth negative advertising about its products, Outtel has its quality control engineers test the speed of 10 Septium 2000 microprocessors as they come off the product line. The test is simply to find the highest clock rate at which each processor works correctly for a selected set of test programs. The maximum clock rates for the 10 processors are as follows (all clock rates are in MHz):

   10, 20, 40, 20, 30, 30, 50, 40, 20, 40

   (a) Find a 99% confidence interval for the maximum clock rate. (24 pts)

$$\overline{X} = \tfrac{1}{10}\sum_{i=1}^{10} X_i = 30$$

$$s^2 = \tfrac{1}{9}\sum_{i=1}^{10}\left(X_i - \overline{X}\right)^2 = 12.4722$$

$$t_9(0.995) = 3.2498$$

The 99% confidence interval computed from these samples is

$$\left(30 - 3.2498 \cdot \sqrt{\tfrac{12.4772}{10}}, 30 - 3.2498 \cdot \sqrt{\tfrac{12.4772}{10}}\right) = (17.1826, 42.8174)$$

(b) How excited will the Outtel marketing people be when they receive the report on the test results from the quality control engineers? (1 pt)

Depends on how much they enjoy job hunting.

Student's t-distribution

The table for Student's t-distribution lists $1-(\alpha/2)$ at the top of each column save the first and the number of degrees of freedom at the left of each row. That is, entry $(n-1, p)$ is $t_{n-1}(1-\frac{\alpha}{2})$.

	$1-\frac{\alpha}{2}$						
$n-1$	0.75	0.9	0.95	0.975	0.99	0.995	0.9995
1	1.0000	3.0777	6.3138	12.7062	31.8207	63.6574	636.6192
2	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248	31.5991
3	0.7649	1.6377	2.3534	3.1824	4.5407	5.8409	12.9240
4	0.7407	1.5332	2.1318	2.7764	3.7649	4.6041	8.6103
5	0.7267	1.4759	2.0150	2.5706	3.3649	4.0322	6.8688
6	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074	5.9588
7	0.7111	1.4149	1.8946	2.3646	2.9980	3.4995	5.4079
8	0.7064	1.3968	1.8595	3.3060	2.8965	3.3554	5.0413
9	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498	4.7809
10	0.6998	1.3722	1.8125	2.2281	2.7638	1.1693	4.5869
11	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058	4.4370
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545	4.3178
13	0.6938	1.3502	1.7709	2.1604	2.6403	3.0123	4.2208
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768	4.1405
15	0.6912	1.3406	1.7531	2.1315	2.6025	2.9467	4.0728
16	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208	4.0150
17	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982	3.9651
18	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784	3.9216
19	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609	3.8834
20	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453	3.8495
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874	3.7251
30	0.6828	1.3104	1.6973	2.0423	2.4573	2.7500	3.6460
35	0.6816	1.3062	1.6896	2.0301	2.4377	2.7238	3.5911
40	0.6807	1.3031	1.6839	2.0211	2.4233	2.7045	3.5510
45	0.6800	1.3006	1.6794	2.0141	2.4121	2.6896	3.5203
50	0.6794	1.2987	1.6759	2.0086	2.4033	2.6778	3.4960
60	0.6786	1.2958	1.6706	2.0003	2.3901	2.6603	3.4602
70	0.6780	1.2938	1.6669	1.9944	2.3808	2.6479	3.4350
80	0.6776	1.2922	1.6641	1.9901	2.3739	2.6387	3.4163
90	0.6772	1.2910	1.6620	1.9867	2.3685	2.6316	3.4019
100	0.6770	1.2901	1.6602	1.9840	2.3642	2.6259	3.3905
∞	0.6754	1.2816	1.6449	1.9600	2.3263	2.5758	3.2905

$t_{n-1}(1-\frac{\alpha}{2})$