

Final Exam Solution

1. A computer system has 3 disks. The system administrator checks the system each day over a long period of time and observes the following. An individual disk that is working one day fails the next day with probability 0.01, independent of the number of other disks that also fail or are already broken. The single technician who services the system can work on only one disk at a time, regardless of how many are broken; hence, at most one disk can get fixed each day. If at least one disk is broken one day, the technician will fix one disk and have it working the next day with probability 0.5. What is the average number of disks that are working? (25 pts)

Solve this problem as a discrete time Markov chain. The state is the number of failed disks. To obtain the state transition probabilities, analyze all possible cases for each state.

| <u>State</u> | <u>Possibilities and associated probabilities</u> |
|---|--|
| 0 (no disks failed) | $\Pr[0 \text{ fail}] = (0.99)^3 = 0.970299$ |
| | $\Pr[1 \text{ fails}] = 3(0.01)(0.99)^2 = 0.029403$ |
| | $\Pr[2 \text{ fail}] = 3(0.01)^2(0.99) = 0.000297$ |
| | $\Pr[3 \text{ fail}] = (0.01)^3 = 0.000001$ |
| 1 (one disk failed) | $\Pr[0 \text{ fail, 0 repaired}] = (0.99)^2(0.5) = 0.49005$ |
| | $\Pr[0 \text{ fail, 1 repaired}] = (0.99)^2(0.5) = 0.49005$ |
| | $\Pr[1 \text{ fails, 0 repaired}] = 2(0.01)(0.99)(0.5) = 0.0099$ |
| | $\Pr[1 \text{ fails, 1 repaired}] = 2(0.01)(0.99)(0.5) = 0.0099$ |
| | $\Pr[2 \text{ fail, 0 repaired}] = (0.01)^2(0.5) = 0.00005$ |
| $\Pr[2 \text{ fail, 1 repaired}] = (0.01)^2(0.5) = 0.00005$ | |
| 2 (two disks failed) | $\Pr[0 \text{ fail, 0 repaired}] = (0.99)(0.5) = 0.495$ |
| | $\Pr[0 \text{ fail, 1 repaired}] = (0.99)(0.5) = 0.495$ |
| | $\Pr[1 \text{ fails, 0 repaired}] = (0.01)(0.5) = 0.005$ |
| | $\Pr[1 \text{ fails, 1 repaired}] = (0.01)(0.5) = 0.005$ |
| 3 (three disks failed) | $\Pr[0 \text{ repaired}] = 0.5$ |
| | $\Pr[1 \text{ repaired}] = 0.5$ |

From these possibilities and their associated probabilities, derive the single step transition probability matrix:

$$\underline{\underline{P}} = \begin{bmatrix} 0.970299 & 0.029403 & 0.000297 & 0.000001 \\ 0.49005 & 0.49995 & 0.00995 & 0.00005 \\ 0 & 0.495 & 0.5 & 0.005 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

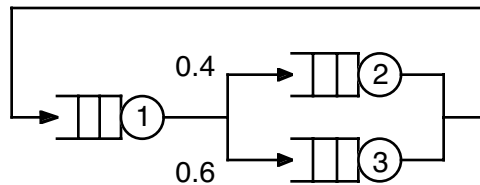
Solve $\underline{\underline{\pi}}\underline{\underline{P}} = \underline{\underline{\pi}}$ plus the normalization equation to obtain

$$\underline{\underline{\pi}} = (0.941211 \quad 0.057045 \quad 0.001719 \quad 0.000025)$$

The expected number of available disks in steady state is

$$E[\text{available disks}] = 3(0.941211) + 2(0.057045) + 1(0.001719) = 2.939442$$

2. The following figure shows a single class queueing network of three queues that has a product form solution.



A job leaving queue 1 goes to queue 2 with probability 0.4 and to queue 3 with probability 0.6. The per visit demands for queues 1, 2, and 3 are 50, 100, and 200, respectively. The network has three jobs in it.

- (a) What is the probability that all of the jobs are in queue 2 and queue 3 (i.e., what is the probability that queue 1 is empty)? You might want to read (b) before deciding on an approach to solving this part. (15 pts)

There are at least two ways to approach this. One is to use MVA to compute the throughput for queue 1 and from that find its utilization. The utilization of queue 1 is the probability that queue 1 has at least one job. One minus the utilization is therefore the probability that queue 1 has no jobs, which is the same as the probability that all of the jobs are in queues 2 and 3.

First find the total demands for the queues.

$$D_1 = 1 \cdot 50 = 50 \quad D_2 = 0.4 \cdot 100 = 40 \quad D_3 = 0.6 \cdot 200 = 120$$

Now apply single class MVA for closed systems:

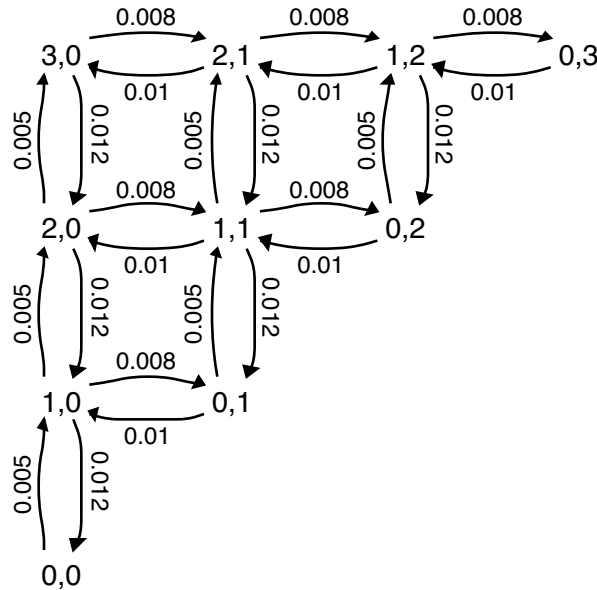
| | N = 0 | N = 1 | N = 2 | N = 3 |
|----------------|-------|--------|--------|--------|
| R ₁ | - | 50 | 61.90 | 70.77 |
| R ₂ | - | 40 | 47.62 | 52.78 |
| R ₃ | - | 120 | 188.6 | 271.8 |
| X | - | 0.0048 | 0.0067 | 0.0076 |
| Q ₁ | 0 | 0.2381 | 0.4153 | 0.5370 |
| Q ₂ | 0 | 0.1905 | 0.3195 | 0.4005 |
| Q ₃ | 0 | 0.5714 | 1.2652 | 2.0625 |

The throughput X for N = 3 is the arrival rate at queue 1. Hence,

$$\rho_1 = 0.0076(50) = 0.38$$

and the probability that all of the jobs are in queues 1 and 2 is $1 - \rho_1 = 0.62$.

This is the correct answer, but will not help you on part (b). An alternative approach, which will also be useful in solving (b), is to analyze the problem using a continuous time Markov chain. Define the state to be (n_1, n_2) , where n_1 is the number of jobs at queue 1 and n_2 is the number of jobs at queue 2. The Markov chain has 10 states, as illustrated in the state transition diagram below:



Using $(3,0), (2,1), (1,2), (0,3), (2,0), (1,1), (0,2), (1,0), (0,1), (0,0)$ as the order of the states, the rate generator matrix for the chain is

$$\underline{\underline{Q}} = \begin{bmatrix} -.02 & .008 & 0 & 0 & .012 & 0 & 0 & 0 & 0 & 0 \\ .01 & -.03 & .008 & 0 & 0 & .012 & 0 & 0 & 0 & 0 \\ 0 & .01 & -.03 & .008 & 0 & 0 & .012 & 0 & 0 & 0 \\ 0 & 0 & .01 & -.01 & 0 & 0 & 0 & 0 & 0 & 0 \\ .005 & 0 & 0 & 0 & -.025 & .008 & 0 & .012 & 0 & 0 \\ 0 & .005 & 0 & 0 & .01 & -.035 & .008 & 0 & .012 & 0 \\ 0 & 0 & .005 & 0 & 0 & .01 & -.015 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .005 & 0 & 0 & -.025 & .008 & .012 \\ 0 & 0 & 0 & 0 & 0 & .005 & 0 & .01 & -.015 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .005 & 0 & -.005 \end{bmatrix}$$

Solve $\underline{\underline{\pi}}\underline{\underline{Q}} = \underline{\underline{0}}$, along with the normalization equation, to get

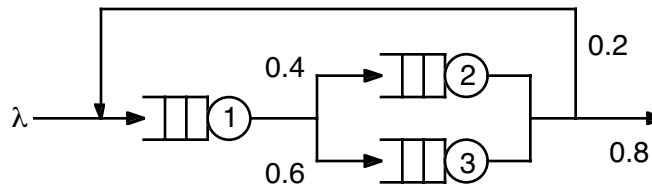
$$\underline{\underline{\pi}} = (.0303 \ .0242 \ .0194 \ .0155 \ .0727 \ .0582 \ .0465 \ .1745 \ .1346 \ .4189)$$

The probability that no jobs are at queue 1 is $\pi_{0,2} + \pi_{0,1} + \pi_{0,0} + \pi_{0,3} = 0.6206$.

(b) What is the probability that the number of jobs in queue 2 and the number of jobs in queue 3 are equal? (10 pts)

Having solved the continuous time Markov chain in (a), we immediately obtain the probability that the numbers of jobs in queues 2 and 3 are equal as $\pi_{1,1} + \pi_{3,0} = 0.0885$.

3. The following figure shows a single class queueing network of three queues that has a product form solution.



Jobs arrive from the outside with rate $\lambda = 1/200$. Jobs leaving queue 1 go to queue 2 with probability 0.4 or to queue 2 with probability 0.6. Jobs leaving queue 2 or queue 3 leave the system with probability 0.8 or go to queue 1 with probability 0.2. The per visit demands for queues 1, 2, and 3 are 50, 100, and 200, respectively.

(a) What is the steady state probability that queues 2 and 3 both have 0 jobs or both have 1 job or both have 2 jobs? (20 pts)

Since this is an open product form queueing network, the joint queue length probability distribution is of the form

$$\Pr[n_1, n_2, n_3] = \rho_1^{n_1} (1 - \rho_1) \rho_2^{n_2} (1 - \rho_2) \rho_3^{n_3} (1 - \rho_3)$$

where n_i and ρ_i are, respectively, the number of jobs in and the utilization of queue i . To obtain the utilizations, find the queue arrival rates.

$$\lambda_2 = 0.4\lambda_1$$

$$\lambda_3 = 0.6\lambda_1$$

$$\lambda_1 = \lambda + 0.2\lambda_1 \Rightarrow \lambda_1 = \lambda/0.8 = 1.25\lambda$$

$$\lambda_2 = 0.5\lambda$$

$$\lambda_3 = 0.75\lambda$$

Since $\lambda = 1/200$, the utilizations are

$$\rho_1 = \lambda_1 \cdot 50 = 0.3125$$

$$\rho_2 = \lambda_2 \cdot 100 = 0.25$$

$$\rho_3 = \lambda_3 \cdot 200 = 0.75$$

Note that all of the queues have utilizations less than one; the network has a steady state solution.

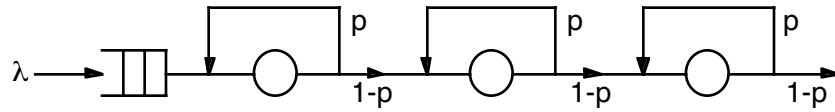
The probability that queues 2 and 3 both have 0 or 1 or 2 jobs is

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \{ \Pr[n_1, 0, 0] + \Pr[n_1, 1, 1] + \Pr[n_1, 2, 2] \} \\ &= \sum_{n_1=0}^{\infty} \rho_1^{n_1} (1 - \rho_1) \{ \rho_2^0 (1 - \rho_2) \rho_3^0 (1 - \rho_3) + \rho_2^1 (1 - \rho_2) \rho_3^1 (1 - \rho_3) + \rho_2^2 (1 - \rho_2) \rho_3^2 (1 - \rho_3) \} \\ &= (1 - \rho_1) (1 - \rho_2) (1 - \rho_3) \sum_{n_1=0}^{\infty} \rho_1^{n_1} \{ \rho_2^0 \rho_3^0 + \rho_2^1 \rho_3^1 + \rho_2^2 \rho_3^2 \} \\ &= (.6875)(.75)(.25) \sum_{n_1=0}^{\infty} \rho_1^{n_1} \{ 1 + (.25)(.75) + (.25)^2 (.75)^2 \} \\ &= (.1289)(1.2227) \frac{1}{1 - \rho_1} = \frac{.1576}{.6875} = .2292 \end{aligned}$$

(b) If the arrival rate is from the outside is doubled, what is the steady state probability that queues 2 and 3 both have 0 jobs or both have 1 job or both have 2 jobs? (5 pts)

The system does not have a steady state, since the utilization of queue 3 would be greater than 1.

4. A system is modeled by the following single queue.



The single server consists of 3 stages that are identical. The one job in service visits each stage in turn. However, each stage may be repeated an arbitrary number of times. The probability that a job that has just received service at a stage repeats the stage is p ; the probability that it goes on to the next stage is $1-p$. Each time a job receives service at a stage, its fixed service time is 1. Hence, a job that visits the first stage just once receives a total service of 1 from the first stage; if it visits the first stage twice (is fed back once), it receives a total service of 2 from the first stage; etc.

If $\lambda = 1/10$ (Poisson arrival process) and $p = 1/3$, what is the average response time (total of service time and waiting time) for the queue?

Let X_i be the random variable for the i th stage, and let $T = X_1 + X_2 + X_3$ be the total service time random variable. We can analyze the system as an M/G/1 queue. To do this, we need the mean and second principal moment of T .

$$E[T] = E[X_1] + E[X_2] + E[X_3] = 3 \cdot E[X_1]$$

$$\begin{aligned} E[X_1] &= \sum_{i=1}^{\infty} i p^{i-1} (1-p) = (1-p) \sum_{i=1}^{\infty} \frac{d}{dp} \{p^i\} \\ &= (1-p) \frac{d}{dp} \left\{ \frac{1}{1-p} - 1 \right\} = \frac{1}{1-p} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} E[(X_1 + X_2 + X_3)^2] &= E[X_1^2] + E[X_2^2] + E[X_3^2] \\ &\quad + 2E[X_1]E[X_2] + 2E[X_1]E[X_3] + 2E[X_2]E[X_3] \\ &= 3E[X_1^2] + 6E[X_1]^2 \end{aligned}$$

$$\begin{aligned} E[X_1^2] &= \sum_{i=1}^{\infty} i^2 p^{i-1} (1-p) = (1-p) \sum_{i=1}^{\infty} \{i(i-1) + i\} p^{i-1} \\ &= (1-p)p \sum_{i=2}^{\infty} i(i-1) p^{i-2} + E[X_1] = (1-p)p \sum_{i=2}^{\infty} \frac{d^2}{dp^2} \{p^i\} + \frac{1}{1-p} \\ &= (1-p)p \frac{d^2}{dp^2} \left\{ \sum_{i=2}^{\infty} p^i \right\} + \frac{1}{1-p} = (1-p)p \frac{d^2}{dp^2} \left\{ \frac{1}{1-p} - 1 - p \right\} + \frac{1}{1-p} \\ &= (1-p)p \cdot \frac{2}{(1-p)^3} + \frac{1}{1-p} = \frac{1+p}{(1-p)^2} \end{aligned}$$

$$E[(X_1 + X_2 + X_3)^2] = 3 \cdot \frac{1+p}{(1-p)^2} + 6 \cdot \frac{1}{(1-p)^2} = \frac{3p+9}{(1-p)^2} = \frac{10}{4/9} = \frac{90}{4}$$

Finally, apply the Pollaczek-Khinchin Mean Value Formula to get the average response time:

$$\bar{R} = \bar{T} + \frac{\lambda \bar{T}^2}{2(1-\rho)} = \frac{9}{2} + \frac{(1/10)(90/4)}{2(1-(9/20))} = \frac{72}{11} \approx 6.55$$