Little's Theorem

Little's Theorem (sometimes called Little's Law) is a statement of what was a "folk theorem" in operations research for many years:

$$\overline{N} = \lambda \overline{T}$$

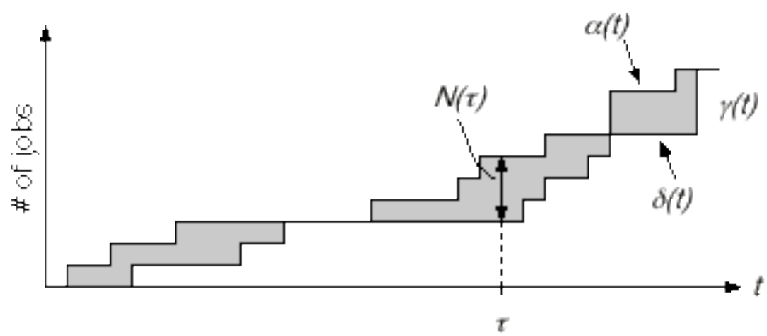
where N is the random variable for the number of jobs or customers in a system, λ is the arrival rate at which jobs arrive, and T is the random variable for the time a job spends in the system (all of this assuming steady-state). What is remarkable about Little's Theorem is that it applies to *any* system, regardless of the arrival time process or what the "system" looks like inside.

Proof:

Define the following:

$$\alpha(t) \equiv \text{ number of arrivals in the interval } (0,t)$$
$$\delta(t) \equiv \text{ number of departures in the interval } (0,t)$$
$$N(t) \equiv \text{ number of jobs in the system at time } t$$
$$= \alpha(t) - \delta(t)$$
$$\gamma(t) \equiv \text{ accumulated customer - seconds in } (0,t)$$

These functions are graphically shown in the following figure:



The shaded area between the arrival and departure curves is $\gamma(t)$.

$$\lambda_t = \text{arrival rate over the interval } (0,t)$$
$$= \frac{\alpha(t)}{t}$$

$\overline{N_t}$ = average # of jobs during the interval $(0,t)$

$$= \frac{\gamma(t)}{t}$$

$\overline{T_t}$ = average time a job spends in the system in $(0,t)$

$$= \frac{\gamma(t)}{\alpha(t)}$$

$$\Rightarrow \quad \gamma(t) = \overline{T_t}\alpha(t)$$

$$\Rightarrow \quad \overline{N_t} = \frac{\overline{T_t}\alpha(t)}{t} = \lambda_t \overline{T_t}$$

Assume that the following limits exist:

$$\lim_{t \to \infty} \lambda_t = \lambda$$

$$\lim_{t \to \infty} \overline{T_t} = \overline{T}$$

Then

$$\lim_{t \to \infty} \overline{N_t} = \overline{N}$$

also exists and is given by $\overline{N} = \lambda \overline{T}$.

Keywords:

Little's Law
Little's Theorem
Steady state