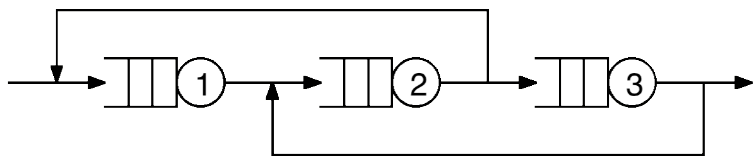
1. In the queueing network shown in the following figure, jobs arrive from the outside world at the rate of 1/12 job per second. The average job service times for queues 1, 2, and 3 are 2, 2, and 5 seconds, respectively. Jobs leaving queue 2 go to queue 1 with probability 0.5 and to queue 3 with probability 0.5. Jobs leaving queue 3 go to queue 2 with probability 0.5 or exit the queueing network with probability 0.5.



   (a) If all three queues are single-server, FCFS queues with exponential service times, what is the average number of jobs in the system?

The routing chain matrix is

$$\underline{\underline{P}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

from which we obtain the following equations ($\lambda_0 = \frac{1}{12}$ is the arrival rate for jobs from outside the system):

$$\begin{aligned} \lambda_0 &= 0.5\lambda_3 && \Rightarrow \quad \lambda_3 = 2\lambda_0 = \tfrac{1}{6} \\ \lambda_3 &= 0.5\lambda_2 && \Rightarrow \quad \lambda_2 = 4\lambda_0 = \tfrac{1}{3} \\ \lambda_1 &= \lambda_0 + 0.5\lambda_2 && \Rightarrow \quad \lambda_1 = 3\lambda_0 = \tfrac{1}{4} \end{aligned}$$

With the throughputs and average service demands, we can compute the utilizations:

$$\begin{aligned} \rho_1 &= \lambda_1 \overline{x_1} = \tfrac{1}{4}(2) = \tfrac{1}{2} \\ \rho_2 &= \lambda_2 \overline{x_2} = \tfrac{1}{3}(2) = \tfrac{2}{3} \\ \rho_3 &= \lambda_3 \overline{x_3} = \tfrac{1}{6}(5) = \tfrac{5}{6} \end{aligned}$$

Since the network is product-form, each queue behaves like an M/M/1 queue in isolation, and the average number of jobs in the system is the sum of the average number of jobs in each queue.

$$\overline{N} = \overline{N_1} + \overline{N_2} + \overline{N_3} = \frac{\rho_1}{1-\rho_1} + \frac{\rho_1}{1-\rho_1} + \frac{\rho_1}{1-\rho_1}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} + \frac{\frac{5}{6}}{1-\frac{5}{6}} = 1 + 2 + 5 = 8$$

(b) What is the probability that there are 2 jobs in each queue if the queues are as in part (a)?

Again, since the network is product-form, the queue occupancy distribution is

$$\pi_{n_1,n_2,n_3} = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\rho_3^{n_3}(1-\rho_3)$$

where n_i is the number of jobs in queue i. Hence,

$$\pi_{2,2,2} = \left(\tfrac{1}{2}\right)^2\left(1-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)^2\left(1-\tfrac{2}{3}\right)\left(\tfrac{5}{6}\right)^2\left(1-\tfrac{5}{6}\right) = \tfrac{25}{11664}$$
$$\approx 0.002143$$

(c) What is the probability that there are 2 jobs in each queue if queues 1 and 3 are single-server, FCFS queues with exponential service times and queue 2 is an infinite server queue?

The only difference between (b) and (c) is the form of the solution for queue 2. The queue occupancy distribution now is

$$\pi_{n_1,n_2,n_3} = \rho_1^{n_1}(1-\rho_1)\frac{\rho_2^{n_2}e^{-\rho_2}}{n_2!}\rho_3^{n_3}(1-\rho_3)$$

$$\Rightarrow \qquad \pi_{n_1,n_2,n_3} = \left(\tfrac{1}{2}\right)^2\left(1-\tfrac{1}{2}\right)\frac{\left(\tfrac{2}{3}\right)^2 e^{-2/3}}{2!}\left(\tfrac{5}{6}\right)^2\left(1-\tfrac{5}{6}\right) \approx 0.001651$$