

Math 101, Spring 2008, Exam 1
Solutions

The exam consists of seven questions. You must show all your work to receive full credit. Please indicate your final answer clearly. Write the honor pledge on your exam when finished. Good luck!

1) Find the derivative with respect to x .

a)

$$\begin{aligned}f(x) &= (x^{10} + 7x^4 - 3x + 3)(\csc x) \\f'(x) &= (10x^9 + 28x^3 - 3)(\csc x) + (x^{10} + 7x^4 - 3x + 3)(-\csc x \cot x)\end{aligned}$$

b)

$$\begin{aligned}y &= x^{\tan x^2} \\ \ln y &= \ln x^{\tan x^2} \\ \ln y &= \tan x^2 \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \sec^2 x^2 \cdot 2x \cdot \ln x + \frac{1}{x} \tan x^2 \\ \frac{dy}{dx} &= x^{\tan x^2} [2x(\sec^2 x^2) \ln x + \frac{1}{x} \tan x^2]\end{aligned}$$

c) $h(x) = e^{x^2 \sin x}$

First, let $g(x) = x^2 \sin x$. $h(x) = e^{g(x)}$, so $h'(x) = e^{g(x)} g'(x)$.

$g'(x) = 2x \sin x + x^2 \cos x$, so $h'(x) = (e^{x^2 \sin x})(2x \sin x + x^2 \cos x)$.

2) Find the equation of the line tangent to the curve $g(x) = \left(\frac{x+1}{x-1}\right)^3$ at the point $(0, -1)$.

First, let $f(x) = \frac{x+1}{x-1}$. $g(x) = (f(x))^3$, so $g'(x) = 3(f(x))^2 f'(x)$.

$f'(x) = \frac{(1)(x-1) - (1)(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$. So $g'(x) = 3\left(\frac{x+1}{x-1}\right)^2 \cdot \frac{-2}{(x-1)^2}$. The slope of the tangent line at $(0, -1)$ is $g'(0) = -6$. Since $-1 = -6(0) + b$, $b = -1$, so the tangent line is $y = -6x - 1$.

3) Find and describe the discontinuities of the function $f(x) = \frac{x^2+5x-14}{x^2-3x+2}$.

$f(x)$ factors into $\frac{(x-2)(x+7)}{(x-2)(x-1)}$. The denominator is zero when $x = 1$ or 2 , so the function is discontinuous at 1 and 2 . Because $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{(x-2)(x+7)}{(x-2)(x-1)} = \lim_{x \rightarrow 2} \frac{x+7}{x-1} = 9$, the discontinuity at $x = 2$ is removable. But as x approaches 1 , the value of the function can get greater than any positive number or less than any negative number (depending on whether you approach from the left or right), so the discontinuity at $x = 1$ is an infinite discontinuity.

4) Use the limit definition of derivative to find the derivative of $g(x) = \frac{x}{x+3}$.

$$\begin{aligned}g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h+3} - \frac{x}{x+3}}{h} \\&= \lim_{h \rightarrow 0} \left(\frac{1}{h} \left[\frac{(x+h)(x+3) - x(x+h+3)}{(x+h+3)(x+3)} \right] \right) \\&= \lim_{h \rightarrow 0} \left(\frac{1}{h} \left[\frac{x^2 + hx + 3x + 3h - x^2 - 3x - hx}{(x+h+3)(x+3)} \right] \right) \\&= \lim_{h \rightarrow 0} \frac{3h}{h(x+h+3)(x+3)} \\&= \lim_{h \rightarrow 0} \frac{3}{(x+h+3)(x+3)} \\&= \frac{3}{(x+3)^2}\end{aligned}$$

5) I want a new soup pot. Create-A-Pot, Inc. charges \$5/in² for copper, \$2/in² for stainless steel, and \$1/in² for glass. I want a one gallon cylindrical pot with a copper bottom, a stainless steel side, and a glass top. The radius of the pot should be between 1 and 10 inches. Keeping in mind the fact that there are approximately 81π cubic inches in a gallon, what is the radius of the pot that minimizes my cost?

A picture is helpful on this one, but I haven't included it in this file. I will let the radius of the pot be r and the height of the pot be h . First, I need to find the cost of each component of the pot.

copper bottom (circular): $5 \cdot \pi r^2$

stainless steel side (rectangular): $2 \cdot 2\pi r h = 4\pi r h$

glass top (circular): $1 \cdot \pi r^2$

So the total cost is $c = 6\pi r^2 + 4\pi r h$. We need to get this all in terms of r . Specifically, we need to get h as a function of r . We know that the total volume of the pot has to be 81π . The volume formula for a cylinder is $V = \pi r^2 h$, so we know that $\pi r^2 h = 81\pi$. Solving for h , we get $h = \frac{81}{r^2}$.

So $c(r) = 6\pi r^2 + 4\pi r \left(\frac{81}{r^2}\right) = 6\pi r^2 + \frac{324\pi}{r}$. For max/min problems, we need to find the critical points, so we need to take the derivative. $c'(r) = 12\pi r - \frac{324\pi}{r^2}$. To find the critical points,

$$\begin{aligned} 0 &= 12\pi r - \frac{324\pi}{r^2} \\ 12\pi r &= \frac{324\pi}{r^2} \\ 12\pi r^3 &= 324\pi \\ r^3 &= \frac{324}{12} \\ r^3 &= 27 \\ r &= 3 \end{aligned}$$

So 3 is the only critical point, but we also need to plug in the two endpoints of the radius interval to see which is a minimum for the cost function. The endpoints are 1 and 10.

$$\begin{aligned} c(1) &= 6\pi(1) + \frac{324\pi}{1} = 330\pi \approx \$1036.73 \\ c(3) &= 6\pi(9) + \frac{324\pi}{3} = 54\pi + 108\pi = 162\pi \approx \$508.94 \\ c(10) &= 6\pi(100) + \frac{324\pi}{10} = 600\pi + 32.4\pi = 632.4\pi \approx \$1986.74 \end{aligned}$$

It's pretty pricey, but the pot with the radius of 3 inches is the best buy. (Of course, you didn't need to put the approximate values down. You could compare the prices just by comparing the coefficients of π .)

6) If $\lim_{x \rightarrow 0^+} f(x) = A$ and $\lim_{x \rightarrow 0^-} f(x) = B$, find the limit.

a) $\lim_{x \rightarrow 0^+} f(x^3 - x^2)$

$\lim_{x \rightarrow 0^+} x^3 - x^2 = 0$, and $x^3 - x^2 < 0$ when x is very close to but greater than 0. (To see this, think about what happens when you evaluate $x^3 - x^2$ for $x = \frac{1}{2}, \frac{1}{3}$, and so on.) If we rename $x^3 - x^2$ as z , $\lim_{x \rightarrow 0^+} f(x^3 - x^2) = \lim_{z \rightarrow 0^-} f(z) = B$.

b) $\lim_{x \rightarrow 0^-} f(x^3 - x^2)$

$\lim_{x \rightarrow 0^-} x^3 - x^2 = 0$, and $x^3 - x^2 < 0$ when x is very close to but less than 0. (To see this, think about what happens when you evaluate $x^3 - x^2$ for $x = \frac{-1}{2}, \frac{-1}{3}$, and so on.) If we rename $x^3 - x^2$ as z , $\lim_{x \rightarrow 0^-} f(x^3 - x^2) = \lim_{z \rightarrow 0^-} f(z) = B$.

7) Use the intermediate value theorem to show that the function $f(x) = 2 + 3 \cos x$ has a root on the interval $[0, \pi]$. What properties of $f(x)$ allow you to use the intermediate value theorem?

You didn't have to quote it on your test, but here's what the intermediate value theorem says: Suppose that the function f is continuous on the closed interval $[a, b]$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is, if K is any number between $f(a)$ and $f(b)$, then there exists at least one number c in (a, b) such that $f(c) = K$.

Thus the property of $f(x)$ that allow us to use the intermediate value theorem is that it is continuous on a closed interval. (We know that because it is just a sum of a constant and a cos function, both of which are continuous everywhere.)

How we use the intermediate value theorem in this case is by noticing that $f(0) = 2 + 3 \cos(0) = 2 + 3 = 5 > 0$ and $f(\pi) = 2 + 3 \cos(\pi) = 2 - 3 = -1 < 0$. So the intermediate value theorem says that there is some k between 0 and π such that $f(k) = 0$.

Bonus: Draw me a picture.

Congratulations, you all got this right!