Math 101, Spring 2008, Practice Exam 2 Solutions

1) a) State the first part of the fundamental theorem of calculus.
Suppose that f is continuous on the closed interval $[a,b]$. If the function F is defined on [a,b] by

$$F(x) = \int_a^x f(t)dt,$$

then F is an antiderivative of f. That is, $F'(x) = f(x)$ for x in $[a,b]$.

b) Use the fundamental theorem of calculus to find the derivative of the function

$$g(x) = \int_{\frac{\pi}{4}}^{x^4} \frac{cos(\sqrt{t})dt}{t^{3/2} + 7t}$$

Let $y = g(x)$ and $u = x^4$. Although we don't need it yet, we might as well compute $\frac{du}{dx} = 4x^3$.

$$y = \int_{\frac{\pi}{4}}^{u} \frac{cos(\sqrt{t})dt}{t^{3/2} + 7t},$$

so

$$\frac{dy}{du} = \frac{\cos(\sqrt{u})}{u^{3/2} + 7u}$$

by the fundamental theorem of calculus. By the chain rule,

$$g'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\frac{\cos(\sqrt{u})}{u^{3/2} + 7u})(4x^3) = \frac{\cos(x^2)}{x^6 + 7x^4} \cdot 4x^3 = \frac{4\cos(x^2)}{x^3 + 7x}.$$

c) Use the fundamental theorem of calculus to find the derivative of the function

$$h(x) = \int_5^x (t^{5000} e^{7t^3})dt$$

This is a more straightforward application of the fundamental theorem of calculus. $h'(x) = x^{5000} e^{7x^3}$.

2)Find the following limits:

a) $\lim_{x\to\infty}(\sqrt{x-7} - \sqrt{x})$

This has the indeterminate form $\infty - \infty$, so we need to multiply by the conjugate to get it into a form we can evaluate.

$$\begin{aligned}
& \lim_{x\to\infty} (\sqrt{x-7} - \sqrt{x}) \\
= & \lim_{x\to\infty} (\sqrt{x-7} - \sqrt{x}) \Big(\frac{\sqrt{x-7} + \sqrt{x}}{\sqrt{x-7} + \sqrt{x}} \Big) \\
= & \lim_{x\to\infty} \frac{x - 7 - x}{\sqrt{x-7} + \sqrt{x}} \\
= & \ 0
\end{aligned}$$

b) $\lim_{x\to 0} \frac{e^x-1}{e^x-e^{-x}}$
This has the indeterminate form $\frac{0}{0}$, so we can use L'Hôpital's rule.

$$\begin{aligned} & \lim_{x\to 0} \frac{e^x-1}{e^x-e^{-x}} \\ = & \lim_{x\to 0} \frac{e^x}{e^x+e^{-x}} \\ = & \frac{1}{1+1} \\ = & \frac{1}{2} \end{aligned}$$

c) $\lim_{x\to 1} \frac{7x}{x-1} - \frac{1}{x^2-1}$
This has the indeterminate form $\infty - \infty$, so we need to get a common denominator (x^2-1)in order to get it into a form we can evaluate.

$$\begin{aligned} & \lim_{x\to 1} \frac{7x}{x-1} - \frac{1}{x^2-1} \\ = & \lim_{x\to 1} \frac{7x(x+1)-1}{x^2-1} \\ = & \lim_{x\to 1} \frac{7x^2+7x-1}{x^2-1} \end{aligned}$$

Since the top is bounded and the bottom number is not, the limit does not exist.

d) $\lim_{x\to 0} x^{\sin x}$ This has the indeterminate form 0^0, so we need to take the natural logarithm to get it into a form we can evaluate.

$$\begin{aligned} L &= \lim_{x\to 0} x^{\sin x} \\ \ln L &= \lim_{x\to 0} \sin x \ln x \end{aligned}$$

This has the indeterminate form $0 \cdot -\infty$, so we need to manipulate it a little more before we can use L'Hôpital's rule.

$$
\begin{aligned}
\ln L &= \lim_{x\to 0} \sin x \ln x \\
&= \lim_{x\to 0} \frac{\ln x}{\csc x} \\
&= \lim_{x\to 0} \frac{\frac{1}{x}}{-\csc x \cot x} \\
&= \lim_{x\to 0} \frac{-\sin x \tan x}{x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} -\tan x \\
&= 1 \cdot 0 \\
\ln L &= 0 \\
e^{\ln L} &= e^0 \\
L &= 1
\end{aligned}
$$

e) $\lim_{x\to\infty} \frac{5x^4-3x^2+7}{-3x^4+7x-2}$
There are two ways to do this. One is to use L'Hôpital's rule several times. The other is to divide the top and bottom by the highest power of x, which in this case is x^4.

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{5x^4-3x^2+7}{-3x^4+7x-2} \\
= & \lim_{x\to\infty} \frac{5-\frac{3}{x^2}+\frac{7}{x^4}}{-3+\frac{7}{x^3}-\frac{2}{x^4}} \\
= & -\frac{5}{3}
\end{aligned}
$$

3) Show that the function $f(x) = x^3$ satisfies the hypotheses of the mean value theorem on the interval $[0, 4]$. Find all numbers c in that interval that satisfy the conclusion of the mean value theorem.

The mean value theorem: Suppose that the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

for some number c in (a, b).

So to answer the question, since $f(x) = x^3$ is continuous on $[0, 4]$, it satisfies the hypotheses of the mean value theorem. To find the numbers c satisfying the conclusion, note that $f(4) = 64$, $f(0) = 0$, and $f'(c) = 3x^2$ for all c.

$$\begin{aligned} 64 - 0 &= 3c^2(4-0) \\ 16 &= 3c^2 \\ \frac{16}{3} &= c^2 \\ \pm\frac{4}{\sqrt{3}} &= c \end{aligned}$$

(If you enjoy rationalizing the denominator, you could also write $c = \pm\frac{4\sqrt{3}}{3}$.) The positive square root is the only one that is in the interval $[0, 4]$, so it is the answer.

4) Find the definite integral:

a)

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx \\ = & \left[2\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} \\ = & 2\tan\frac{\pi}{4} - 2\tan 0 \\ = & 2(1) - 2(0) \\ = & 2 \end{aligned}$$

b)

$$\begin{aligned} & \int_1^4 \frac{dx}{x^2} \\ = & \left[-\frac{1}{x}\right]_1^4 \\ = & -\frac{1}{4} - (-1) \\ = & \frac{3}{4} \end{aligned}$$

c)

$$\begin{aligned} & \int_2^4 \frac{4dx}{x} \\ = & \left[4\ln x\right]_2^4 \\ = & 4(\ln 4 - \ln 2) \\ = & 4(\ln \frac{4}{2}) \\ = & 4\ln 2 \end{aligned}$$

d)

$$\begin{aligned} & \int_0^{\ln 4} 3e^{\frac{x}{2}}\,dx \\ = & \left[6e^{\frac{x}{2}}\right]_0^{\ln 4} \\ = & 6e^{(\frac{1}{2})\ln 4} - 6e^0 \\ = & 6e^{\ln 2} - 6(1) \\ = & 6(2) - 6 \\ = & 6 \end{aligned}$$

5) Consider the function $f(x) = x^2 + x + 1$ on the interval $I = [0, 5]$.

a) Using a partition of I into 5 regular subintervals, use a right-endpoint sum to compute the Riemann sum for $f(x)$.

The width of each subinterval is 1, and the right endpoints of the subintervals will be 1, 2, 3, 4, and 5. The values of $f(x)$ at those points are 3, 7, 13, 21, and 31, respectively. So the Riemann sum associated to this partition and selection is $3 + 7 + 13 + 21 + 31 = 75$.

b) Without completing part c, is this an overestimate or an underestimate of the integral?

Since the function is always increasing on the interval $[0, 5]$, the right endpoint sum is an overestimate. (To see that the function is always increasing on $[0, 5]$, note that the derivative is $2x + 1$, which is always positive on this interval.)

c) Using the Riemann sum definition of the integral, find $\int_0^5 f(x)dx$.

We can choose any sequence of partitions we want (theorem 2 in section 5.4), so we will choose a sequence of regular partitions and the right-endpoint sum. Using formulas (15) and (16) in section 5.4, note that for such partitions and selections, $[0, 5]$, $\Delta x = \frac{5}{n}$ and $x_i = 0 + \frac{5i}{n}$. We don't need it yet, but $f(x_i) =$

$(\frac{5i}{n})^2 + \frac{5i}{n} + 1.$

$$\begin{aligned}
\int_0^5 f(x)dx &= \lim_{n\to\infty} \Sigma_{i=1}^n f(x_i)\Delta x \\
&= \lim_{n\to\infty} \Sigma_{i=1}^n \left[((\frac{5i}{n})^2 + \frac{5i}{n} + 1)\cdot\frac{5}{n}\right] \\
&= \lim_{n\to\infty} \left[\Sigma_{i=1}^n \frac{125i^2}{n^3} + \Sigma_{i=1}^n \frac{25i}{n^2} + \Sigma_{i=1}^n \frac{5}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{125}{n^3}\Sigma_{i=1}^n i^2 + \frac{25}{n^2}\Sigma_{i=1}^n i + 5\right] \\
&= \lim_{n\to\infty} \left[\frac{125(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n)}{n^3} + \frac{25(\frac{1}{2}n^2 + \frac{1}{2}n)}{n^2} + 5\right] \\
&= \frac{125}{3} + \frac{25}{2} + 5
\end{aligned}$$

Note: The formulas for $\Sigma_{i=1}^n i^2$ and $\Sigma_{i=1}^n i$ are from section 5.3. If you need to use any of them other than $\Sigma_{i=1}^n i$ on the test, I will provide them.

6) a) Find $\frac{dy}{dx}$ when y is defined implicitly by the equation $2y = x^2 + \sin y$.

$$\begin{aligned}
2y &= x^2 + \sin y \\
2\frac{dy}{dx} &= 2x + \cos y\frac{dy}{dx} \\
(2 - \cos y)\frac{dy}{dx} &= 2x \\
\frac{dy}{dx} &= \frac{2x}{2 - \cos y}
\end{aligned}$$

b) Find the first two derivatives of y with respect to x when y is defined implicitly by the equation $2x^3 - 3y^2 = 7$.

$$\begin{aligned}
2x^3 - 3y^2 &= 7 \\
6x^2 - 6y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{6x^2}{6y} = \frac{x^2}{y}
\end{aligned}$$

To find the second derivative, we just take the derivative of the first derivative.

$$\begin{aligned}\frac{dy}{dx} &= \frac{6x^2}{6y} = \frac{x^2}{y} \\ \frac{d^2y}{dx^2} &= \frac{y(2x) - x^2\frac{dy}{dx}}{y^2} \\ &= \frac{2xy - x^2(\frac{x^2}{y})}{y^2} \\ &= \frac{2x}{y} - \frac{x^4}{y^3}\end{aligned}$$

7) Let $f(x) = x^4 + 2x^3$.

a) Find and classify all critical points of the graph of $y = f(x)$.

$f(x)$ is defined for all x, so we only need to find points where $f'(x) = 0$. $f'(x) = 4x^3 + 6x^2 = 2x^2(2x+3)$. This is 0 when $x = 0$ or $-\frac{3}{2}$, so these are the critical points. $f''(x) = 12x^2 + 12x = 12x(x+1)$. $f''(-\frac{3}{2}) > 0$, so $f(-\frac{3}{2})$ is a local minimum. Since $f''(0) = 0$, we can't use the second derivative test to see whether $f(0)$ is a local extremum. Instead, notice that $2x^2$ is always positive, and near 0, $2x+3$ is positive. So $f'(x)$ does not change sign at 0, so $f(0)$ is not a local extremum.

b) Where is $f(x)$ increasing and decreasing? The derivative is negative on the interval $(-\infty, -\frac{3}{2})$, so f is decreasing there. The derivative is positive on the interval $(-\frac{3}{2}, +\infty$, so f is increasing there.

c) Sketch a graph of $f(x)$. Label local extrema and inflection points. Be sure to include behavior "at infinity" and any asymptotes.

I will draw a graph in class on Friday. As shown in part a, the only local extreme value is $f(-\frac{3}{2}) = -\frac{27}{16}$, which is a local (and global) minimum. The inflection points are $(0,0)$ and $(-1,-1)$. (The second derivative changes sign at both $x = 0$ and $x = -1$.) The function is concave up on the interval $(-\infty, -1)$, concave down on the interval $(-1, 0)$, and concave up on the interval $0, +\infty$. Because the leading term is x^4, the function increases without bound as x approaches either ∞ or $-\infty$. There are no asymptotes.

8)

$$\begin{aligned}f(x) &= \frac{x^2-3}{2x-4} \\ f'(x) &= \frac{2(x-3)(x-1)}{(2x-4)^2} \\ f''(x) &= \frac{8}{(2x-4)^3}\end{aligned}$$

Use this information to sketch a graph of x. Label local extrema and inflection points. Be sure to include behavior "at infinity" and any asymptotes.

I will draw a graph in class on Friday.

Asymptotes: By polynomial long division, we can write $f(x) = \frac{x}{2} + 1 + \frac{1}{2x-4}$. $\lim_{x\to\pm\infty} f(x) = \frac{x}{2} + 1$, so the line $y = \frac{x}{2} + 1$ is a slant asymptote. The function is undefined at $x = 2$. We can see that $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, so the line $y = 2$ is a vertical asymptote.

Critical points: The derivative is defined for all x such that $f(x)$ is defined, so the only critical points are where $f'(x) = 0$. This happens when $x = 3$ or 1. Since $f''(3) > 0$ and $f''(1) < 0$, $f(3) = 3$ is a local minimum and $f(1) = 1$ is a local maximum.

Concavity: $f''(x) < 0$ when $x < 2$, and $f''(x) > 0$ when $x > 2$. There are no inflection points because the function is not defined at $x = 2$.

9) A 13-foot ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

Note: I would strongly recommend that you draw a picture to help you answer these questions. You should also name your variables. I called the distance from the house to the base of the ladder l and the height where the ladder hit the house h.

a) How fast is the top of the ladder sliding down the wall then?

The Pythagorean Theorem states that $h^2 + l^2 = 13^2$. (We don't need it yet, but note that when $l = 12$, $h = 5$.) Taking the derivative of both sides with respect to time, we get $2h\frac{dh}{dt} + 2l\frac{dl}{dt} = 0$. Since we know l, h, and $\frac{dl}{dt}$, we can solve for $\frac{dh}{dt}$.

$$\begin{aligned} 2h\frac{dh}{dt} + 2l\frac{dl}{dt} &= 0 \\ 10\frac{dh}{dt} + 24(5) &= 0 \\ \frac{dh}{dt} &= \frac{-120}{10} = -12 \end{aligned}$$

The ladder is sliding down the wall at a rate of 12 ft/sec. (Note: I would accept either 12 or -12. The sign just says whether you are thinking of up or down as the "forward" or "positive" direction.)

b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

The area of the triangle is $A = \frac{1}{2}hl$. Taking the derivative of both sides with respect to time, $\frac{dA}{dt} = \frac{1}{2}h\frac{dl}{dt} + \frac{1}{2}l\frac{dh}{dt}$. Since we know all of the quantities on the right side (we found $\frac{dh}{dt}$ in part a), we can easily substitute in to get $\frac{dA}{dt} = \frac{1}{2}(5\cdot 5 + 12\cdot(-12)) = \frac{1}{2}(25 - 144) = \frac{1}{2}(-119) = -59.5$. The area of the triangle is decreasing at the rate of 59.5 ft^2/sec.

c) At what rate is the angle θ between the ladder and the ground changing then?

We can find an expression for θ by noticing that $\frac{h}{13} = \sin\theta$, or $h = 13\sin\theta$. Taking the derivative of both sides with respect to time, we get that $\frac{dh}{dt} = 13\cos\theta\frac{d\theta}{dt}$. We know $\frac{dh}{dt} = -12$. $\cos\theta = \frac{l}{13}$. At the time in question, $l = 12$, so $\cos\theta = \frac{12}{13}$. Putting it all together, we get that $-12 = 13(\frac{12}{13})(\frac{d\theta}{dt}$, so $\frac{d\theta}{dt} = -1$. The angle between the ladder and the ground is decreasing at the rate of 1 rad/sec.

10) Find the indefinite integral:

a) $\displaystyle\int(x^3 - 3x + 7)dx = \frac{x^4}{4} - \frac{3x^2}{2} + 7x + C$

b) $\displaystyle\int \csc^2(\pi x)dx = -\frac{\cot\pi x}{\pi} + C$

c) $\displaystyle\int \frac{1}{3x}dx = \frac{\ln x}{3} + C$

d) $\displaystyle\int 5e^{-\frac{x}{2}}dx = -10e^{-\frac{x}{2}} + C$

e) $\displaystyle\int(3\sin 2x - 2\cos 3x)dx - \frac{3}{2}\cos 2x - \frac{2}{3}\sin 3x + C$

f) $\displaystyle\int \frac{1}{(2x+1)^2}dx = -\frac{1}{2(2x+1)} + C$