Math 101 Spring 2008 Practice Test 1 Solutions

1) Find the derivative with respect to x.
Note: For problems like this, you don't necessarily have to simplify as much as I did.

a) $y = x^{x^2}$

$$\begin{aligned} y &= x^{x^2} \\ \ln y &= \ln x^{x^2} \\ &= x^2 \ln x \\ \frac{1}{y}\frac{dy}{dx} &= (x^2)(\frac{1}{x}) + 2x \ln x \\ &= x + 2x \ln x \\ \frac{dy}{dx} &= x^{x^2}(x + 2x \ln x). \end{aligned}$$

b) $f(x) = (3x^2 + 7x - 10)(csc(x^2))$

$$\begin{aligned} f(x) &= (3x^2 + 7x - 10)(csc(x^2)) \\ f'(x) &= (6x + 7)(\csc x^2) + (3x^2 + 7x - 10)(-\csc x^2 \cot x^2 \cdot 2x) \\ &= (6x + 7)(\csc x^2) - (2x)(3x^2 + 7x - 10)(\csc x^2 \cot x^2) \end{aligned}$$

c) $g(x) = \sin(\sqrt[3]{x})$

$$\begin{aligned} g(x) &= \sin(\sqrt[3]{x}) \\ g'(x) &= \cos(\sqrt[3]{x}) \cdot \frac{1}{3} x^{-\frac{2}{3}} \\ &= \frac{\cos(\sqrt[3]{x})}{3\sqrt[3]{x^2}} \end{aligned}$$

d) $y = x \ln \frac{x+1}{x-1}$

$$\begin{aligned} y &= x \ln \frac{x+1}{x-1} \\ &= x[\ln(x+1) - \ln(x-1)] \\ y' &= 1 \cdot \ln \frac{x+1}{x-1} + x(\frac{1}{x+1} - \frac{1}{x-1}) \\ &= \ln \frac{x+1}{x-1} - \frac{2x}{x^2 - 1} \end{aligned}$$

e) $v(x) = (x - \frac{1}{x})^7$

$$\begin{aligned} v(x) &= (x - \frac{1}{x})^7 \\ v'(x) &= 7(x - \frac{1}{x})^6 \cdot (1 + \frac{1}{x^2}) \end{aligned}$$

f) $f(x) = \sin e^x \cos e^x$

$$\begin{aligned} f(x) &= \sin e^x \cos e^x \\ f'(x) &= (\cos e^x)(\cos e^x)(e^x) + (\sin e^x)(-\sin e^x)(e^x) \\ &= e^x(\cos^2 e^x - \sin^2 e^x) \\ &= e^x(\cos 2e^x) \end{aligned}$$

2) Use the limit definition of derivative to find the derivative.

a) $y = \frac{1}{\sqrt{x}}$

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h\to 0} (\frac{1}{h})(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{(x)(x+h)}}) \\ &= \lim_{h\to 0} (\frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x^2+hx})})(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}) \\ &= \lim_{h\to 0} \frac{x - x - h}{h(\sqrt{x^2+hx})(\sqrt{x} + \sqrt{x+h})} \\ &= \frac{-1}{\sqrt{x^2} \cdot 2\sqrt{x}} \\ &= \frac{-1}{2} x^{\frac{-3}{2}} \end{aligned}$$

b) $f(x) = x^2 - x + 2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) + 2 - (x^2 - x + 2)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x - h + 2 - x^2 + x - 2}{h} \\ &= \lim_{h\to 0} \frac{h(2x - 1 + h)}{h} \\ &= \lim_{h\to 0} 2x - 1 + h \\ &= 2x - 1 \end{aligned}$$

3) Evaluate the limit if it exists.

a)

$$\begin{aligned}
& \lim_{t\to 3} \frac{t^2-9}{t^2-5t+6} \\
= & \lim_{t\to 3} \frac{(t+3)(t-3)}{(t-3)(t+2)} \\
= & \lim_{t\to 3} \frac{t+3}{t-2} \\
= & \frac{\lim_{t\to 3} t+3}{\lim_{t\to 3} t-2} \\
= & \frac{6}{1} \\
= & 6
\end{aligned}$$

b)

$$\begin{aligned}
& \lim_{x\to 0} x \cot 3x \\
= & \lim_{x\to 0} \frac{x\cos 3x}{\sin 3x} \\
= & \lim_{x\to 0} \frac{3x\cos 3x}{3\sin 3x} \\
= & \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \lim_{x\to 0} \frac{cos 3x}{3} \\
= & 1 \cdot \frac{1}{3} \\
= & \frac{1}{3}
\end{aligned}$$

4) Use the intermediate value theorem to show that the function $y = 3x^3 + 3x^2 - 3x - 1$ has 3 roots in the interval $[-2, 2]$.
When $x = -2$, $y = 3(-8) + 3(4) - 3(-2) - 1 = -7$.
When $x = -1$, $y = 3(-1) + 3(1 - 3(-1) - 1 = 2$. Because y is continuous for all x, the intermediate value theorem states that $y = 0$ for some x between -2 and -1. Continuing this way, we see that when $x = 0$, $y = -1$, and when $x = 1$, $y = -2$. So there is a root between -1 and 0 and another between 0 and 1.

5) State the intermediate value theorem and use it to show that every real number has a cube root.
The intermediate value theorem: Suppose that the function f is continuous on the closed interval $[a, b]$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is, if K is any number between $f(a)$ and $f(b)$, then there exists at least one number c in (a, b) such that $f(c) = K$.
To show that every real number has a cube root, first observe that $0^3 = 0$,

$1^3 = 1$, and $(-1)^3 = -1$. Let a be a real number other than 0, 1, or -1. We can't yet assume $\sqrt[3]{a}$ exists yet. But consider the function $f(x) = x^3 - a$. $f(a) = a^3 - a = a(a^2 - 1)$. $f(-a) = -a^3 - a = -a(a^2 - 1) = -(f(a))$. Since $a \neq 0, -1$, or 1, one of these values is positive and one is negative, so $x^3 - a = 0$ has a solution in the interval $[-a, a]$ (or $[a, -a]$ if a is negative). So every real number has a cube root. QED.

6) If $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$, find the limit.

a)$\lim_{x\to 0^+} f(x^3 - x)$

$\lim_{x\to 0^+} x^3 - x = 0$, and $x^3 - x < 0$ when x is very close to but greater than 0. (To see this, think about what happens when you evaluate $x^3 - x$ for $x = \frac{1}{2}, \frac{1}{3}$, and so on.) So $\lim_{x\to 0^+} f(x^3 - x) = \lim_{x\to 0^-} f(x) = B$.

b) $\lim_{x\to 0^-} f(x^3 - x)$

$\lim_{x\to 0^-} x^3 - x = 0$, and $x^3 - x > 0$ when x is very close to but less than 0. (Think about what happens when you evaluate $x^3 - x$ for $x = \frac{-1}{2}, \frac{-1}{3}$, and so on.) So $\lim_{x\to 0^-} f(x^3 - x) = \lim x \to 0^+ f(x) = A$.

c) $\lim_{x\to 0^+} f(x^2 - x^4)$

$\lim_{x\to 0^+} x^2 - x^4 = 0$ and $x^2 - x^4 > 0$ when x is very close to 0, whether positive or negative. (As usual, think about what happens when $x = \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}$, etc.) So $\lim_{x\to 0^+} f(x^2 - x^4) = \lim_{x\to 0^+} f(x) = A$.

d) $\lim_{x\to 0^-} f(x^2 - x^4)$

$\lim_{x\to 0^-} x^2 - x^4 = 0$, and as noted in part (c), $x^2 - x^4 > 0$ when x is very close to 0, so $\lim_{x\to 0^-} f(x^2 - x^4) = \lim_{x\to 0^+} f(x) = A$.

7) You are designing a **rectangular** poster containing a 50 in^2 **rectangle** of printing with margins of 4 inches each at top and bottom and 2 inches at each side. What overall dimensions will minimize the amount of paper used? **You may assume that the poster's width is between** 5 **and** 54 **inches.**

This is an applied optimization problem. We are trying to find the minimum value of a function on a closed interval. (That's why I had to add the assumption that the width was between 5 and 54 inches.) To do this, we have to figure out where the value of the derivative of the function is 0. Note: Drawing a picture will help you solve this problem, but I did not include one in these solutions.

First, I chose variables. I chose x for the height of the rectangle of printing and y for its width. You could also choose the variables to be the height and width of the poster itself, but I found that my choice led to easier computations. Taking into account the margins, the height of the poster is $x + 8$ inches, and its width is $y + 4$ inches. We want to minimize area, which can be written $A = (x+8)(y+4)$. We also know that the area of the printing is $xy = 50$. So we can say that $y = \frac{50}{x}$. Putting this all together, we want to minimize the function $A(x) = (x + 8)(\frac{50}{x} + 4) = 4x + \frac{400}{x} + 82$. Taking the derivative with respect to x, we get $\frac{dA}{dx} = 4 - \frac{400}{x^2}$. $\frac{dA}{dx} = 0$ when $4 = \frac{400}{x^2}$, or when $4x^2 = 400$, so $x = \pm 10$. $x = -10$ doesn't make sense for a height, so $x = 10$ is the only critical point we need to test. We also need to test the endpoints of the values of x. Since the endpoints I specified in the problem were for the width of the overall poster, and x is the height of the rectangle of text, I need to see what those two values would do to my x value. If the width of the poster were 5 inches, that would mean $y = 1$, so $x = 50$. If the width of the poster were 50 inches, that would make $y = 50$, so $x = 1$. So we need to check the area when $x = 1, 10$, and 50. $A(1) = (9)(54) = 486$ in^2. $A(10) = (18)(9) = 162$ in^2. $A(50) = (58)(5) = 540$ in^2. $x = 10$ is clearly the minimum. So the poster should be 18 inches tall and 9 inches wide.

8) Find and describe the discontinuities of the function $f(x) = \frac{x^2-9}{x^2-5x+6}$.
$f(x) = \frac{(x+3)(x-3)}{(x-3)(x-2}$. This is undefined at $x = 3, -2$. But as we saw in problem (3a), $\lim_{x \to 3} f(x) = 6$, so $f(x)$ has a removable discontinuity at $x = 3$. But $\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x+3}{x-2}$, which does not exist. In fact, that number can be arbitrarily large in either the positive or negative sense by plugging in numbers very close to -2 on either side. So $f(x)$ has an infinite discontinuity at $x = -2$.

9) Find the equation of the line tangent to the curve $y = \sqrt{x^2 + e^x - e}$ when $x = 1$.
Using the chain rule, we get $\frac{dy}{dx} = \frac{1}{2}(x^2+e^x-e)^{\frac{-1}{2}} \cdot (2x+e^x)$. We want to find the equation of the tangent line when $x = 1$. The slope is given by the derivative, so the slope of the tangent line at $x = 1$ is $\frac{1}{2}(1 + e - e)^{\frac{-1}{2}} \cdot (2 + e) = 1 + \frac{e}{2}$. When $x = 1$, $y = (1 + e - e)^{\frac{1}{2}} = 1$, so we want the line through the point $(1, 1)$ with slope $1 + \frac{e}{2}$. $1 = (1 + \frac{e}{2})(1) + b$, so $b = \frac{-e}{2}$. The equation of the tangent line is $y = (1 + \frac{e}{2})x - \frac{e}{2}$.

10) Below is a table of values of two functions and their derivatives at various points. Let $h(x) = (f \circ g)(x)$ and $k(x) = (g \circ f)(x)$.

x	f(x)	f'(x)	g(x)	g'(x)
-3	-2	0	1	-2
-2	-1	1	3	2
-1	0	3	0	-1
0	1	3	0	1
1	2	2	1	-1
2	3	-1	-2	3
3	-3	-2	-2	0

a) Does the graph of $h(x)$ have a horizontal tangent line **at any of the x values listed in the table**? If so, find the equation of that line (or lines).
To determine whether/where a function has a horizontal tangent, we have to determine whether/where its derivative is 0. Using the chain rule, $h'(x) = f'(g(x))(g'(x)$. This is a product, so it is 0 when either $f'(g(x)) = 0$ or $g'(x) = 0$. Using the table, $f'(g(x)) = 0$ when $g(x) = -3$. But $g(x)$ is never -3, so there is no x that makes that part of the product 0. $g'(x) = 0$ when $x = 3$, so $h'(3) = 0$. Since the tangent line is horizontal, we know it has the form $y = c$ for some constant c. To find $c = h(3)$ because the tangent line goes through the point

$3, h(3)$. $h(3) = f(g(3)) = f(-2) = -1$. So the equation of the tangent line is $y = -1$.

b) Does the graph of $k(x)$ have a horizontal tangent line **at any of the x values listed in the table**? If so, find the equation of that line (or lines).

This is very similar to part (a). In this case, $k'(x) = g'(f(x))f'(x)$, so we want to find values of x such that $g'(f(x)) = 0$ or $f'(x) = 0$. $g'(f(x)) = 0$ when $f(x) = 3$. $f(x) = 3$ when $x = 2$, so $k'(2) = 0$. $k(2) = g(f(2)) = g(3) = -2$, so the equation for the tangent line at this point is $y = -2$. $k'(x)$ is also 0 when $f'(x) = 0$. This happens when $x = -3$. $k(-3) = g(f(-3)) = g(-2) = 3$, so the equation of the tangent line is $y = 3$.

c) Is $h(x)$ increasing, decreasing, or neither at $x = 2$?

$h'(x) = f'(g(x))g'(x)$, so $h'(2) = f'(g(2))g'(2) = f'(-2)(3) = 1 \cdot 3 = 3$. Since this is positive, $h(x)$ is increasing at $x = 2$.

11) Let $f(x) = x^2 + 3x + 4$ if $x \geq 1$ and $f(x) = x^3 + 3x + 4$ if $x < \mathbf{1}$.

a) Is $f(x)$ continuous for all x?

When $a < 1$, $\lim_{x\to a} f(x) = \lim_{x\to a} x^3 + 3x + 4 = a^3 + 3a + 4 = f(a)$, so f is continuous at a. When $a > 1$, $\lim_{x\to a} f(x) = \lim_{x\to a} x^2 + 3x + 4 = a^2 + 3a + 4 = f(a)$, so f is continuous at a. To determine whether $f(x)$ is continuous at $x = 1$, we have to take the right- and left-hand limits. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 3x + 4 = 1 + 3 + 4 = 8 = f(1)$. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 + 3x + 4 = 1 + 3 + 4 = 8$. Since these limits agree and equal $f(1)$, the function is continuous at 1.

b) Is $f(x)$ differentiable for all x?

We apply the same procedure for the derivative of $f(x)$. If $x \neq 1$, the derivative is continuous at x because it is just a polynomial in x. When $x = 1$, we take the right- and left-hand limits of the derivative. $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} 2x + 3 = 5$. $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 3x^2 + 3 = 6$. Since these limits do not agree, the function is not differentiable at 1.