Math 101 Spring 2008 Review for the final-solutions

1) Find the limit if it exists.

a) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4}$

$\frac{5}{4}$

b) $\lim_{x\to 0} \frac{\sin 5x}{x}$

This is $\lim_{x\to 0} \frac{5\cdot\sin 5x}{5x} = 5$.

c) $\lim_{x\to 0} \frac{1-\cos 3x}{2x^2}$

This has indeterminate form 0/0. You can use L'Hôpital's rule or multiply by the conjugate of the top. I used L'Hôpital's rule and got $\lim_{x\to 0} \frac{1-\cos 3x}{2x^2} = \lim_{x\to 0} \frac{3\sin 3x}{4x} = \lim_{x\to 0} \frac{9\cos 3x}{4} = \frac{9}{4}$

d) $\lim_{x\to 0^-} \frac{|5x|}{x}$

-5

e) $\lim_{x\to 0} x^3 \cos \frac{1}{x}$

$-1 \le \cos \frac{1}{x} \le 1$, so $-x^3 \le x^3 \cos \frac{1}{x} \le x^3$. Since $\lim_{x\to 0} -x^3 = 0 = \lim_{x\to 0} x^3$, by the squeeze law of limits, $\lim_{x\to 0} x^3 \cos \frac{1}{x} = 0$.

f) $\lim_{x\to 0} \sin\left(\frac{\pi}{2}(\frac{\sin x}{x})\right)$

Let $g(x) = \frac{\pi}{2} \cdot \frac{\sin x}{x}$ and $f(u) = \sin u$. $\lim_{x\to 0} g(x) = \frac{\pi}{2}$, and $\lim_{u\to\frac{\pi}{2}} f(u) = 1$. So $\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} \sin\left(\frac{\pi}{2}(\frac{\sin x}{x})\right) = 1$.

g) $\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$

This has the indeterminate form 0/0. By L'Hôpital's rule, this is $\lim_{x\to 0} \frac{e^{\sin x}\cdot\cos x}{1} = 1$

h) $\lim_{x\to\infty} (\ln 2x - \ln(x+1))$

Using laws of logarithms, $\ln 2x - \ln(x+1) = \ln \frac{2x}{x+1}$. Let $g(x) = \frac{2x}{x+1}$. Then $\lim_{x\to\infty} g(x) = 2$. So $\lim_{x\to\infty} \ln\left(\frac{2x}{x+1}\right) = \lim_{g(x)\to 2} \ln(g(x)) = \ln 2$.

i) $\lim_{x\to\infty} x^{1/\ln x}$

Let

$$\begin{aligned}
L &= \lim_{x\to\infty} x^{1/\ln x} \\
\ln L &= \ln x^{1/\ln x} \\
\ln L &= \frac{1}{\ln x} \cdot \ln x \\
\ln L &= 1 \\
L &= e^1 = e
\end{aligned}$$

2) Use the epsilon-delta definition of the limit to prove that $\lim_{x\to 7} 2x - 3 = 11$. Given any $\epsilon > 0$, we want to find a $\delta > 0$ such that if $|x-7| < \delta$, then $|(2x-3) - 11| < \epsilon$. (In simpler terms, that means that if x is within δ of 7, $2x-3$ is within ϵ of 11.) Working backwards a bit, here's a little manipulation

we can do:

$$\begin{aligned}|(2x-3)-11| &< \epsilon\\ |2x-14| &< \epsilon\\ 2|x-7| &< \epsilon\\ |x-7| &< \frac{\epsilon}{2}\end{aligned}$$

Wow, $|x-7|$ appeared earlier! Now I have found my δ. My manipulations tell me that if $|x-7| < \frac{\epsilon}{2}$, then $|(2x-3)-11| < \epsilon$. So $\delta = \frac{\epsilon}{2}$. Now to actually prove it:
Let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{2}$. Then if

$$\begin{aligned}|x-7| &< \delta\\ |x-7| &< \frac{\epsilon}{2}\\ 2|x-7| &< \epsilon\\ |2x-14| &< \epsilon\\ |(2x-3)-11| &< \epsilon\end{aligned}$$

This means that if you give me any positive number, I can figure out how close x must be to 7 in order to get $2x-7$ within that amount of 11. For example, if you had given me the positive number $\frac{1}{2}$, I would know that if $6\frac{3}{4} < x < 7\frac{1}{4}$, then $10\frac{1}{2} < 2x-7 < 11\frac{1}{2}$. I could have figured that out using other methods, but since I found a formula for δ in terms of ϵ, I can automatically spit out that bound given any positive number at all.

3) Find and describe any discontinuities of the function.

a) $f(x) = \frac{|5x|}{x}$

$\lim_{x\to 0} f(x)$ does not exist (the right-hand limit is 5 and the left-hand limit is -5), so the function is discontinuous at $x = 0$. The discontinuity is not removable. It is a finite jump discontinuity.

b) $f(x) = \frac{1}{x-2} - 3x$

The function is discontinuous at $x = 2$. The discontinuity is not removable. It is an infinite discontinuity, also known as a vertical asymptote.

c) $f(x) = \frac{x^2+3x-10}{x-2}$

The function is discontinuous at $x = 2$. It is a removable discontinuity because $\lim_{x\to 2} f(x) = 7$.

4) Use the intermediate value theorem to show that there exists a number that is exactly one less than its cube.
Let a be that number if it exists. We want to show that it has to exist. If it does, it satisfies the equation $a^3 - a = 1$, or $a^3 - a - 1 = 0$. Figuring out whether such a number exists is the same as asking whether the function $f(x) = x^3 - x - 1$ has a real root (or whether $f(x) = 0$ for any x). $f(1) = -1 < 0$, and $f(2) = 5 > 0$. By the intermediate value theorem, since f is continuous, $f(a) = 0$ for some a between 1 and 2. This is the desired number.

5) a) Use the limit definition of the derivative to find the derivative of the function $g(x) = 4 - x^2$.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{4-(x+h)^2-(4-x^2)}{h} \\
&= \lim_{h\to 0} \frac{4-(x^2+2xh+h^2)-4+x^2}{h} \\
&= \lim_{h\to 0} \frac{-2xh+h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2x+h)}{h} \\
&= \lim_{h\to 0} -2x+h = -2x
\end{aligned}$$

b) Find the equation of the tangent line to the curve when $x = 2$.
When $x = 2$, $g'(x) = -2x = -4$. $g(2) = 4 - 4 = 0$, so the tangent line has slope $m = -4$ and passes through the point $(2, 0)$. The line has the equation $y = mx + b$, so $0 = -4(2) + b$, so $b = 8$. The tangent line is $y = -4x + 8$.

6) Find the derivative of the function with respect to x.

a) $F(x) = \int_{\sin x}^{7} t dt$ $F(x) = \int_{\sin x}^{7} t dt = -\int_{7}^{\sin x} t dt$. Let $u = \sin x$. $\frac{du}{dx} = \cos x$. Let $y(u) = F(u(x))$. So $y(u) = -\int_{7}^{u} t dt$. By the fundamental theorem of calculus, $\frac{dy}{du} = -u$. By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin x \cos x$.

b) $y = \sqrt{x(x+1)}$
$y = \big(x(x+1)\big)^{1/2} = (x^2+1)^{1/2}$. So $\frac{dy}{dx} = \frac{1}{2}(x^2+x)^{-1/2} \cdot (2x+1) = \frac{2x+1}{2\sqrt{x^2+x}}$

c) $g(x) = \frac{x^5-3x^3+\pi x-7}{x-2}$
Massive fun with the quotient rule! I'm not going all the way through it. The final answer is:

$$\frac{(x-2)(5x^4-9x^2+\pi)-(x^5-3x^3+\pi x-7)}{(x-2)^2} = \frac{4x^5-10x^4-6x^3+18x^2+7-2\pi}{(x-2)^2}$$

You could, of course, leave it in the form on the left.

d) $y = (\sin x)^{\cos x}$
I used logarithmic differentiation.

$$\begin{aligned}
\ln y &= \cos x \ln \sin x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\cos x}{\sin x} \cdot \cos x - \sin x \ln \sin x \\
\frac{dy}{dx} &= y\Big[\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\Big] \\
&= (\sin x)^{\cos x}\Big[\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\Big]
\end{aligned}$$

As an aside, Note that since there is a natural log of sine in there, the derivative is only defined when $\sin x > 0$. That is OK, though, because the original function only makes sense when $\sin x > 0$. There are many intervals on which this function could be defined. One is $0 < x < \pi$.

e) $f(x) = e^{x^2} \sin x$

$f'(x) = e^{x^2} \cos x + 2xe^{x^2} \sin x$

7) Below is the graph of the curve $y = \sqrt{\cos x}$ on one interval where it is defined.

a) Determine the interval shown.

The interval is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. You can tell because it has to be an interval where $\cos x$ is positive, and it includes both positive and negative x values.

b) Find the volume obtained by revolving this segment of the curve about the x-axis.

Since it is a function of y in terms of x, and we are revolving it around the x-axis, we should use the cross section method.

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \pi(\sqrt{\cos x})^2 dx \\ &= \int_{-\pi/2}^{\pi/2} \pi(\cos x) dx \\ &= \pi\Big[\sin x\Big]_{-\pi/2}^{\pi/2} \\ &= \pi(1 - (-1)) \\ &= 2\pi \end{aligned}$$

8) Below is the graph of the curve $y = \sin x$ from $x = 0$ to $x = \pi$. Set up, but do not solve, the following integrals.

a) The volume obtained by revolving the region bounded by that curve and the x-axis about the x-axis.

$V = \displaystyle\int_0^\pi \pi(\sin x)^2\, dx = \int_0^\pi \pi \sin^2 x\, dx$

b) The volume obtained by revolving the region bounded by that curve and the x-axis about the y-axis.

$V = \displaystyle\int_0^\pi 2\pi x \sin x dx$

c) The length of the curve.

$\frac{dy}{dx} = \cos x$

$S = \displaystyle\int_0^\pi \sqrt{1 + \cos^2 x}\, dx$

d) The surface area obtained by revolving the curve about the x-axis.

$A = \displaystyle\int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x}\, dx$

e) The surface area obtained by revolving the curve about the y-axis.

$A = \displaystyle\int_0^\pi 2\pi x \sqrt{1 + \cos^2 x}\, dx$

9) Consider the region in the first quadrant of the plane bounded by the curves

$x = y$ and $x = y^3$.

a) Find the area of the region.

The curves intersect in the first quadrant when $y = y^3$, which happens when $y = 0, 1$. Between 0 and 1, $y \geq y^3$, so the area is given by the integral

$\int_0^1 (y - y^3)dy = \left[\frac{y^2}{2} - y^4 4\right]_0^1 = \frac{1}{4}$

b) Find the volume of the solid obtained by revolving this region about the x-axis.

$V = \int_0^1 2\pi y(y - y^3)dy = \int_0^1 2\pi(y^2 - y^4)dy = 2\pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi(\frac{1}{3} - \frac{1}{5}) = \frac{4\pi}{15}$

c) Find the volume of the solid obtained by revolving this region about the y-axis.

$V = \int_0^1 \pi(y - y^3)^2 dy = \int_0^1 \pi(y^2 - y^6)dy = \pi\left[\frac{y^3}{3} - \frac{y^7}{7}\right]_0^1 = \pi(\frac{1}{3} - \frac{1}{7}) = \frac{4\pi}{21}$

10) Here is some information about a function $f(x)$.

x	f(x)	f'(x)	f"(x)
-1	15	-4	36
0	10	0	0
1	7	-8	-24
2	6	-16	0
3	-17	0	36

Assuming $f(x)$ and its first two derivatives are continuous and there are no other zeroes in the first two derivatives, sketch a graph of $f(x)$. Label any local extrema and inflection points.

The graph has a horizontal tangent at $x = 0, 3$. At $x = 0$, it is not a local extreme value. At $x = 3$, there is a local minimum. There are inflection points at $x = 0, 2$. The graph is concave up and decreasing when $x < 0$. It is concave down and decreasing when $0 < x < 2$. It is concave up and decreasing when $2 < x < 3$. It is concave up and increasing when $x > 3$.

11) Sketch a graph of the function $y = \frac{x^2-1}{x}$

There is a vertical asymptote at $x = 0$, and the line $y = x$ is a slant asymptote. There are no critical points or points of inflection. The function is increasing and concave up when $x < 0$ and decreasing and concave down when $x > 0$.

12) A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

As usual with this type of problem, a picture is very helpful. Mine is a right triangle. The bottom edge is 500 (the distance between the lift-off point and the range finder), and I called the other leg (not the hypotenuse) y. I named the angle opposite to y as θ.

We know from this picture that $500 \tan\theta = y$. By implicitly differentiating both sides with respect to time t, $500 \sec^2\theta \frac{d\theta}{dt} = \frac{dy}{dt}$. At the pertinent time,

$\theta = \pi/4$, so $\sec^2\theta = 2$. We are also told in the problem that $\frac{d\theta}{dt} = 0.14$. So $\frac{dy}{dt} = 500 \cdot 2 \cdot 0.14 = 140$ ft/min.

13) Find two positive numbers whose sum is 20 and whose product is as large as possible.
Let x and y be the numbers. We know $x + y = 20$ and both are positive, so the minimum value of $x = 0$ and the maximum value of $x = 20$. We want to maximize $p(x) = xy = x(20-x)$. $p'(x) = 20 - 2x = 2(10-x)$. The only critical point is $x = 10$. We need to check the endpoints and the point $x = 10$ to see which is the maximum. $p(0) = 0$, $p(10) = 100$, $p(20) = 0$, so the numbers are $x = y = 10$, and their product is 100.

14) A police officer saw my car going 57 mph in Italy, TX at 4 pm. Two hours later, another officer saw my car going 45 mph in Paris, TX, 150 miles away. The speed limit on the highway was 60 mph. What theorem from calculus did the second officer use to give me a ticket?
The mean value theorem! If we let $f(t)$ be my distance in miles from Italy at the time t hours, my velocity is $f'(t)$. It is pretty safe to assume that my distance from Italy was a continuous function. $f(0) = 0$ and $f(2) = 150$. The mean value theorem states that since $f(t)$ is continuous, there exists some time c between 0 and 2 such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{150}{2} = 75$$

Since $f'(t)$ can be interpreted as my velocity, at some point between 4 pm and 6 pm I had to be going exactly 75 mph, which was well above the speed limit.

15) Find the indefinite integral.

a) $\displaystyle\int z^3(1+z^4)^3 dz$

$\frac{(1+z^4)^4}{16} + C$

b) $\displaystyle\int \frac{\sqrt{x}}{3} + \frac{3}{\sqrt{x}} dx$

$\frac{2x^{3/2}}{9} + 6\sqrt{x} + C$

c) $\displaystyle\int x^2 e^{3x^3-9x} - e^{3x^3-9x} dx$

The integral equals $\displaystyle\int (x^2-1)(e^{3x^3-9x})dx$, so we use the substitution $u = e^{3x^3-9x}$; $du = (9x^2-9)e^{3x^3-9x}$

$$\int (x^2-1)(e^{3x^3-9x})dx = \frac{1}{9}\int du = \frac{1}{9}u + C = \frac{1}{9}e^{3x^3-9x} + C$$

16) The velocity of a particle is given by the equation $v(t) = t^2 - 5t + 6$. It travels from time $t = 0$ to time $t = 4$.

a) Find its net displacement.

$$\int_0^4 t^2 - 5t + 6\, dt = \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^4 = \frac{64}{3} - 16 = 5\frac{1}{3}$$

b) Find its total displacement.

Total displacement is $\int |v(t)|dt$, so we need to figure out when the velocity is positive and when it is negative. $t^2 - 5t + 6 = (t-2)(t-3)$, so it is positive when $0 < t < 2$, negative when $2 < t < 3$, and positive when $3 < t < 4$. So the total displacement is

$$\begin{aligned} & \int_0^2 t^2 - 5t + 6\,dt + \int_2^3 -(t^2 - 5t + 6)\,dt + \int_3^4 t^2 - 5t + 6\,dt \\ = & \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^2 - \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_2^3 + \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_3^4 \\ = & 4\frac{2}{3} - \frac{-1}{6} + \frac{5}{6} = 5\frac{2}{3} \end{aligned}$$

17) Find the definite integral.

a) $\displaystyle\int_9^4 \frac{1-\sqrt{x}}{\sqrt{x}}dx$

The integral can be rewritten as

$\displaystyle\int_9^4 x^{-1/2} - 1\,dx = \left[2x^{1/2} - x\right]_9^4 = 4 - 4 - (6 - 9) = 3$

b) $\displaystyle\int_0^{\pi/3} 4\sec y \tan y\,dy$

$\left[4\sec y\right]_0^{\pi/3} = 4(2) - 4 = 4$

c) $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2 + \tan\frac{t}{2}\right)\sec^2\frac{t}{2}dt$

Let $u = 2 + \tan\frac{t}{2}$; $du = \frac{1}{2}\sec^2\frac{t}{2}dt$; $u(-\pi/2) = 1, u(\pi/2) = 3$.

$\displaystyle\int_{-\pi/2}^{\pi/2} \left(2 + \tan\frac{t}{2}\right)\sec^2\frac{t}{2}dt = \int_1^3 2u\,du = \left[u^2\right]_1^3 = 8$

d) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}dx$

Let $u = x^4 + 9$; $du = 4x^3dx$; $u(0) = 9, u(1) = 10$.

$\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}dx = \int_9^{10} \frac{1}{4}u^{-1/2}du = \left[\frac{1}{2}u^{1/2}\right]_9^{10} = \frac{\sqrt{10}}{2} - \frac{3}{2}$

18) $xy = \sin x + \cos y$. Find the derivative of y with respect to x, assuming y is defined implicitly as a function of x.

Differentiating both sides with respect to x, we get

$$\begin{aligned} y + x\frac{dy}{dx} &= \cos x - \sin y\frac{dy}{dx} \\ (x + \sin y)\frac{dy}{dx} &= \cos x - y \\ \frac{dy}{dx} &= \frac{\cos x - y}{x + \sin y} \end{aligned}$$

19) If $\frac{dy}{dt} = 12t(3t^2 - 1)^3$ and $y(1) = 3$, find y as a function of t.

This is an initial value problem. Since $\frac{dy}{dt} = 12t(3t^2-1)^3$, we know that $y(t)$ is an antiderivative of $12t(3t^2-1)^3$, so we will integrate that function. Using $u = 3x^2 - 1$, we get
$y(t) = \displaystyle\int 12t(3t^2-1)^3 dt = \int 2u^3 du = \frac{u^4}{2} + C = \frac{(3t^2-1)^4}{2} + C$. Now we need to find C. Since $y(1) = 3$, $3 = \frac{(3-1)^4}{2} + C = 8 + C$, so $C = -5$.

20) Use the Riemann sum definition of the integral to compute $\displaystyle\int_1^3 4x - 2dx$.

For any regular partition of the interval $(1,3)$ into n subintervals, we will have $\Delta x = \frac{2}{n}$, $x_i = 1 + \frac{2i}{n}$. Substituting into our Riemann sum definition of the integral, we get

$$\begin{aligned}
\int_1^3 4x - 2dx &= \lim_{n\to\infty} \Sigma_{i=1}^n f(x_i)\Delta x \\
&= \lim_{n\to\infty} \Sigma_{i=1}^n \left(4(1+\frac{2i}{n}) - 2\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \Sigma_{i=1}^n \frac{4}{n} + \lim_{n\to\infty} \Sigma_{i=1}^n \frac{16i}{n} \\
&= \lim_{n\to\infty} 4 + \lim_{n\to\infty} \Sigma_{i=1}^n \frac{16(n^2+n)}{2n^2} \\
&= 4 + 8 = 12
\end{aligned}$$

I have not included any questions on 6.7-6.9, which will be bonus on the test. If you want to practice those sections, do the 6.7-6.9 optional assignment.