Challenge Problem 3

Problem: Let q be an integer. Prove that if $f(x) = x^{\frac{1}{q}}$,$f'(x) = \frac{1}{q}x^{\frac{-(q-1)}{q}}$.

Solution: Remember the algebraic formula

$$(s^q - t^q) = (s - t)(s^{q-1} + s^{q-2}t + ... + st^{q-2} + t^{q-1}).$$

If we substitute $x^{\frac{1}{q}}$ for s and $a^{\frac{1}{q}}$ for t, we get

$$(x - a) = (x^{\frac{1}{q}} - a^{\frac{1}{q}})(x^{\frac{q-1}{q}} + x^{\frac{q-2}{q}}a^{\frac{1}{q}} + ... + x^{\frac{1}{q}}a^{\frac{q-2}{q}} + a^{\frac{q-1}{q}}). \tag{1}$$

Using the definition of derivative,

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^{\frac{1}{q}} - a^{\frac{1}{q}}}{x - a}.$$

Now we can use (1) to get

$$f'(x) = \lim_{x\to a} \frac{x^{\frac{1}{q}} - a^{\frac{1}{q}}}{(x^{\frac{1}{q}} - a^{\frac{1}{q}})(x^{\frac{q-1}{q}} + x^{\frac{q-2}{q}}a^{\frac{1}{q}} + ... + x^{\frac{1}{q}}a^{\frac{q-2}{q}} + a^{\frac{q-1}{q}})}.$$

This becomes

$$f'(x) = \lim_{x\to a} \frac{1}{(x^{\frac{q-1}{q}} + x^{\frac{q-2}{q}}a^{\frac{1}{q}} + ... + x^{\frac{1}{q}}a^{\frac{q-2}{q}} + a^{\frac{q-1}{q}})}.$$

Evaluating the limit, we get $f'(x) = \frac{1}{qx^{\frac{q-1}{q}}}$, or $f'(x) = \frac{1}{q}x^{\frac{-(q-1)}{q}}$.