

**1. *Why introduce money into our models?***

- There are several issues we are interested in that can only be discussed in a monetary economy:
  - (i) We want to discuss inflation as a macroeconomic phenomenon in its own right and separately from either growth or business cycles. Since the nominal price level is nothing but the rate of exchange between money and goods and services it is clear we need to introduce money into our model in order to study inflation.
  - (ii) We are also interested in the interactions between money and other variables. It is a centuries-old idea that inflation stimulates economic activity. This can be seen in the writings of Hume and the quantity theorists (it is worthwhile looking at Hume's essay). Also note that the tendency of nominal prices to move pro-cyclically in pre-WWII data, and the tendency of the rate of inflation to vary pro-cyclically in at least some cycles in post WWII data, are consistent with the idea that money shocks are a causal factor in some business cycle fluctuations.

**2. *The relationship between money and other variables***

- The graphs we previously examined showed that there appeared to be some relationship in the post-war US data between changes in money growth and subsequent movements in real GDP growth. However, the relationship also appears to have weakened in recent decades as money growth rates have become more variable while GDP has become less so. Over longer periods of time, money growth rates are more closely related to inflation rates than to real GDP growth rates. The graphs are repeated below for convenience.
- Apart from the recent US data, international evidence also suggests that severe monetary shocks, particularly "currency crises" and severe contractions in money growth rates, can have substantial real effects. There is perhaps a hint in the evidence both from the current developed economies in earlier eras as well as currently underdeveloped economies, that monetary shocks have larger real effects when financial markets are less developed.

Chart 1

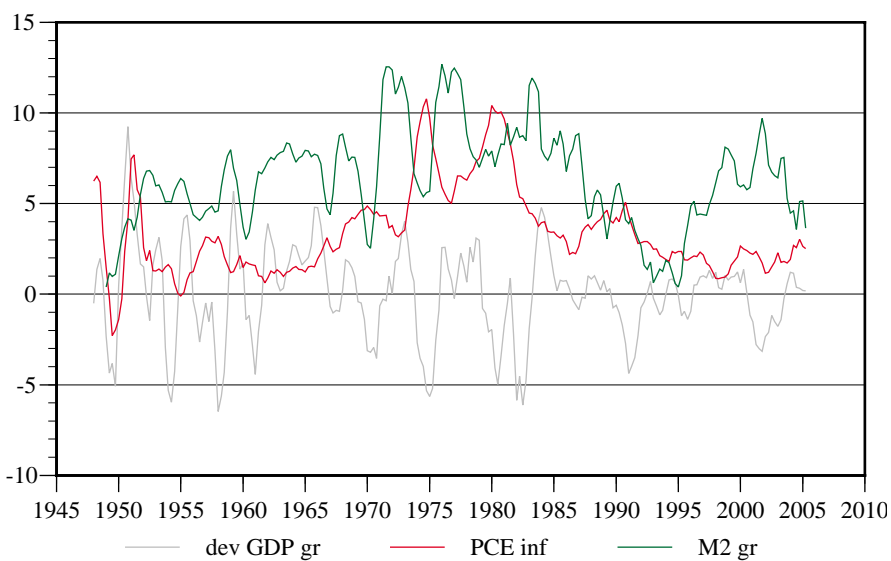
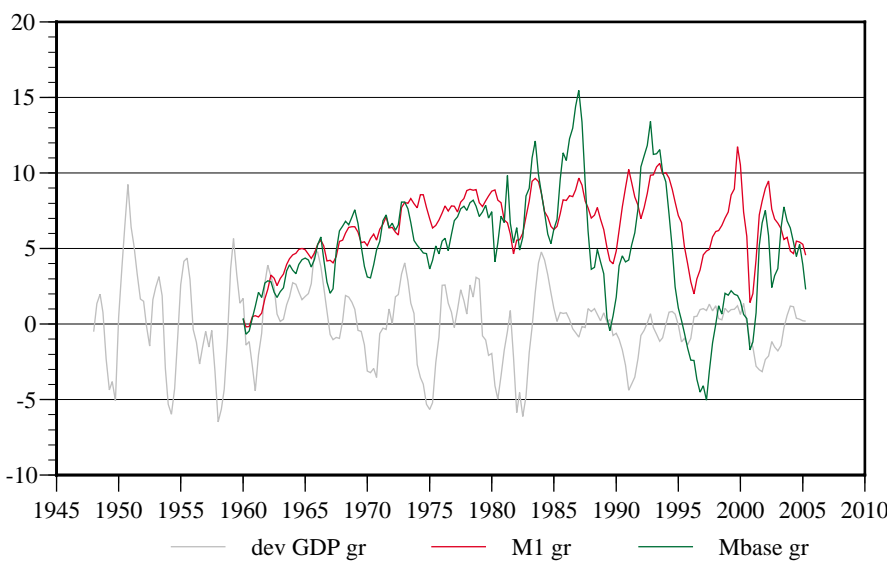


Chart 2



### 3. Money in the Cass Koopmans Model

- A monetary analogue to the Cass-Koopmans model is the model considered by Sidrauski. A representative consumer has preferences:

$$\sum \beta^t U(c_t, m_t) \quad (1)$$

where  $m_t$  represents real money balances. It is assumed that the transaction services provided by money depend on the rate of exchange between money and goods. If the quantity of nominal balances doubles but the nominal price of goods also doubles, then the transaction services supplied by the money are unchanged. Thus real balances  $m$  appear in the utility function.

- Technology is as in the Cass-Koopmans model:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad (2)$$

with new money received as cash transfers of real value:

$$v_t = \frac{\tilde{M}_{t+1} - \tilde{M}_t}{P_t} \quad (3)$$

where  $P_t$  denotes the price of the output in terms of money.

- The individual budget constraint now becomes:

$$P_t(c_t + k_{t+1}) + M_{t+1} = P_t [f(k_t) + (1-\delta)k_t] + M_t + P_t v_t \quad (4)$$

with the values of  $M$  *chosen* by the individual to be distinguished from the values of  $\tilde{M}$ , the per capita money *supplied* by the government. In equilibrium, money demand will equal money supply so an equilibrium condition will be:

$$\tilde{M}_t = M_t \quad (5)$$

but we cannot use this condition to cancel out  $M$  and  $v$  from the individual budget constraint. We assume the individual maximizes utility by choosing money balances and consumption and we want equilibrium prices to be such that he chooses to hold the quantity of money supplied.

- Now if we define  $m_t = M_t/P_t$  and  $\pi_t = (P_{t+1} - P_t)/P_t$  we get:

$$M_{t+1}/P_t = (M_{t+1}/P_{t+1})(P_{t+1}/P_t) = m_{t+1}(1 + \pi_t) \quad (6)$$

and the budget constraint can be written in real terms as:

$$c_t + k_{t+1} + m_{t+1}(1 + \pi_t) = f(k_t) + (1 - \delta)k_t + m_t + v_t \quad (7)$$

- Assume the representative individual has perfect foresight so that all future  $\pi_\tau$  and  $v_\tau$  are known at time  $t < \tau$ . As in the Cass–Koopmans model we introduce Lagrange multipliers  $\beta^t q_t$  for the budget constraint at time  $t$ . Specifically, we define the Lagrangian:

$$L = \sum \beta^t \{ U(c_t, m_t) + q_t [f(k_t) + (1 - \delta)k_t + m_t + v_t - c_t - k_{t+1} - m_{t+1}(1 + \pi_t)] \} \quad (8)$$

and set equal to zero the derivatives of  $L$  with respect to  $c_t$ ,  $k_{t+1}$ , and<sup>1</sup>  $m_{t+1}$  to get (for  $t \geq 0$ ):

$$0 = U_c(c_t, m_t) - q_t \quad (9)$$

$$0 = \beta^{t+1} U_m(c_{t+1}, m_{t+1}) + q_{t+1} \beta^{t+1} - \beta^t q_t (1 + \pi_t) \quad (10)$$

$$0 = \beta^{t+1} q_{t+1} [f'(k_{t+1}) + (1 - \delta)] - \beta^t q_t \quad (11)$$

$$c_t + k_{t+1} + m_{t+1}(1 + \pi_t) = f(k_t) + (1 - \delta)k_t + m_t + v_t \quad (12)$$

with initial conditions  $k_0$ ,  $M_0$  and a TVC for  $q$ .

- From these first order conditions we get difference equations in  $q$ ,  $k$  and  $m$  which are solved using the two initial conditions and the transversality condition. The solution will depend on the path assumed for the exogenous money supply process  $v_t$ .

#### 4. *Stationary States*

- We can study the stationary point of the set of difference equations by looking for a set of values

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<sup>1</sup>Since the individual takes the price level as given choosing real balances is equivalent to choosing nominal balances.

of the capital stock, real money balances and shadow price of capital  $q$  which remain constant over time. It is clear that there is no hope of getting a stationary solution for an arbitrary money supply process  $\{v_t\}$ . However, suppose we have  $v_t = v$  is constant (that is, the real change in the money supply is constant). Then we must have

$$v_t = M_{t+1}/P_t - M_t/P_t = m_{t+1}(1 + \pi_t) - m_t = v \quad (13)$$

and at a stationary point  $m_{t+1} = m_t = m$  so that  $v = m\pi$ . Then the first order conditions (9), (10), (11) and (12) become:

$$U_c(c, m) = q \quad (14)$$

$$\beta U_m(c, m) + \beta q = q(1 + \pi) \quad (15)$$

$$\beta[f'(k) + (1-\delta)] = 1 \quad (16)$$

$$c = f(k) - \delta k \text{ using } v = m\pi \quad (17)$$

Hence,  $f'(k) + 1 - \delta = 1/\beta = 1 + \rho$  for  $\rho$  the rate of time discounting or

$$f'(k) = \rho + \delta \quad (18)$$

so that stationary  $k$  depends only on technology and preferences as in the Cass-Koopmans model. Given the stationary value of  $k$  we obtain the stationary value of  $c$  from  $c = f(k) - \delta k$ . To find the stationary value for real balances we use the (14) and (15) to get:

$$U_c(\rho + \pi + \rho\pi) = U_m \quad (19)$$

which, with  $v = m\pi$ , is solved for the stationary values of real money balances and inflation given  $v$  and  $c$ . Specifically, if we substitute  $\pi = v/m$  into (19) we get

$$U_m = U_c(\rho + (1+\rho)v/m) \quad (20)$$

which can be solved for  $m = m(v)$ . Note that we can interpret (19) as saying that the ratio of the marginal utility of real money balances to the marginal utility of consumption in a stationary state must equal a *nominal* rate of interest - the rate of time discount  $\rho$  plus the rate of inflation  $\pi$  plus an interaction term  $\rho\pi$ . In stationary equilibrium, there is no interaction between the monetary and real sectors. A change in the real rate of money supply increase  $v$  will affect the stationary level of real money balances and the stationary inflation rate but will not affect the *stationary* values of  $c$  or  $k$ . Also observe that the demand for money is a demand for real money balances - a once-over change in the quantity of money supplied will merely produce a once-over increase in the nominal price level to restore equilibrium between the demand and supply of money.

- Now observe that, comparing stationary states with different rates of real money growth  $v$ , we have

$$\frac{d\pi}{dv} = \frac{1}{m} - \frac{vm'(v)}{m^2}. \quad (21)$$

We can then use (20) to determine  $m'(v)$ :

$$m'(v) = \frac{U_c \frac{1+\rho}{m}}{U_{mm} - \left[ \rho + \frac{(1+\rho)v}{m} \right] U_{cm} + \frac{(1+\rho)v}{m^2} U_c} = \frac{U_c(1+\rho)m}{\left\{ U_{mm} - \frac{U_m U_{cm}}{U_c} \right\} m + (1+\rho)v U_c} \quad (22)$$

The term in braces  $\{ \}$  is negative if we assume neither  $c$  nor  $m$  is an inferior good.<sup>2</sup>

- Thus, starting from  $v$  near 0,  $m'(v) < 0$  and increases in  $v$  will decrease stationary money demand and increase the stationary rate of inflation. As  $v$  is increased,  $m(v)$  decreases. Suppose we as-

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<sup>2</sup>. That is, if we consider the imaginary problem of choosing the two goods  $c$  and  $m$  to maximize  $U(c, m)$  subject to  $p_c c + p_m m = y$  and impose the requirement that  $dc/dy > 0$  and  $dm/dy > 0$  then we must have  $U_{mm} - U_m U_{cm} / U_c < 0$ .

sume that at some level of  $v$ , say  $\bar{v}$ , the denominator in (22) becomes zero, while it remains negative for all  $v < \bar{v}$ . Then we would find  $m'(v) \rightarrow -\infty$  as  $v \rightarrow \bar{v}$ . However, if  $m(0) > 0$ , there must then be some level of  $v < \bar{v}$  such that  $m = 0$  and hence, from (22),  $m'(v) = 0$ , which contradicts the original assumption. Thus, we conclude that we must have  $m \rightarrow 0$ , and  $m'(v) \rightarrow 0$ , as  $v$  increases. Furthermore, as  $m \rightarrow 0$ ,  $\pi = v/m$  increases without bound. Stationary equilibrium inflation rates enter into the hyper-inflationary range and people do what they can to avoid using money to finance their transactions.

### 5. *Behavior out of stationary states*

- While the stationary values of  $c$  and  $k$  are independent of  $v$ , there will in general be an interaction between the real and monetary sectors of the economy out of the stationary state. This results from changes in real money balances altering the marginal utility of consumption, as can be seen if we consider the special additively-separable utility function:

$$U(c, m) = W(c) + Z(m). \quad (23)$$

In this case, the marginal utility of consumption is independent of  $m$ , and under the policy  $v_t = v$  (constant over time) the first order conditions (9), (10), (11) and (12) become:

$$W'(c_t) = q_t \quad (24)$$

$$Z'(m_t) = -q_t + (1 + \pi_{t-1})q_{t-1}/\beta \quad (25)$$

$$q_t [f'(k_t) + (1 - \delta)] = q_{t-1}/\beta \quad (26)$$

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t \quad (27)$$

Equations (24), (26) and (27) are just the Cass-Koopmans equations which are solved for  $c_t$  and  $k_t$  independently of monetary policy.

- For the additively separable utility case, we can obtain the demand for money for periods prior

to long run stationary equilibrium from (25):

$$m_t = (Z')^{-1} [(1+\pi_{t-1})q_{t-1}/\beta - q_t] \quad (28)$$

Then  $Z$  concave implies  $(Z')^{-1} \equiv G(\cdot)$  has  $G' < 0$ . Now re-write the argument of  $G$  as:

$$(1+\pi_{t-1})q_{t-1}/\beta - q_t = q_{t-1} [1+\rho+\pi_{t-1}+\rho\pi_{t-1}-q_t/q_{t-1}] = q_{t-1} [\rho+(1+\rho)\pi_{t-1}-(q_t-q_{t-1})/q_{t-1}] \quad (29)$$

The demand for money depends negatively on a nominal interest rate and the shadow value of capital  $q_{t-1}$ . The term in square brackets is a nominal interest rate since  $(q_t-q_{t-1})/q_{t-1}$  is the capital gain on real capital  $k$  and  $\rho$  is a rate of time preference so  $\rho - (q_t-q_{t-1})/q_{t-1}$  is a real interest rate and  $(1+\rho)\pi_{t-1}$  is an inflation premium. Since  $G' < 0$ , the demand for money falls as the nominal interest rate rises. The term  $q_{t-1}$  falls as wealth  $k$  increases so the demand for money rises as real wealth increases.

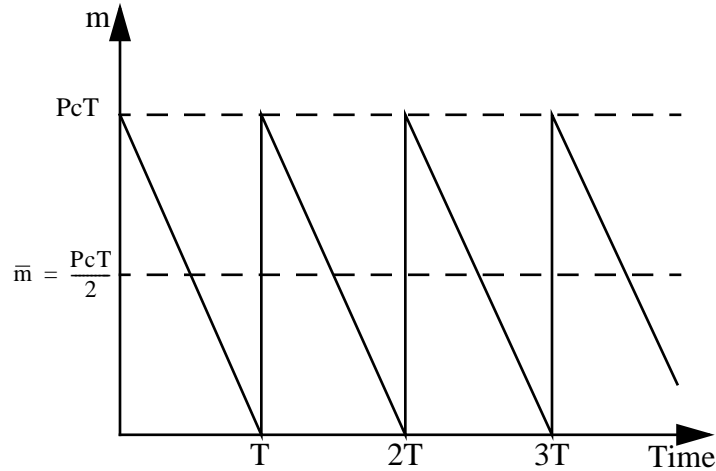
## 6. *Transactions Model of Money Demand*

- Barro discusses a simple model of the transactions demand for money balances that I believe you have already studied in your previous macroeconomics courses. However, it may be worth briefly reviewing the model here. Those students unfamiliar with the model should carefully read the relevant chapters in Barro.
- Consider an individual who wishes to use money to finance a flow of transactions, uniformly distributed over time. Instead of holding non-interest bearing money, this individual could hold bonds yielding a *nominal* interest rate of  $R_t$ , which are, however, unacceptable as a medium of exchange. The bonds can be converted to money in order to finance transactions, but it costs the individual a *real* amount  $\gamma$  for each conversion.<sup>3</sup> Following a conversion, the money balances will be exhausted gradually over time. Let  $T$  be the time interval (fraction of the unit time inter-

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<sup>3</sup>.Barro defines transactions costs as a *nominal* amount  $\gamma$ . However, since these costs are likely to involve the expenditure of labor time more than a nominal withdrawal fee, it makes more sense to define them in real terms.

val) between conversions of bonds to money. Assume the desired expenditure is an instantaneous flow at rate  $Pc$ . The total money balances required to fund expenditure for an interval of length  $T$  will then be  $PcT$ . The holdings of money balances will then follow a saw-toothed path as in the following diagram:



The number of conversions *per period* will be  $1/T$ , and this will impose a *real* transactions cost of

$$\gamma/T \tag{30}$$

The average money balances under this scenario will be

$$\bar{m} = \frac{PcT}{2} \tag{31}$$

resulting in foregone interest earnings of real value

$$\frac{R\bar{m}}{P} = R\frac{cT}{2} \tag{32}$$

- The total cost  $\tau$  of financing transactions using money is therefore

$$\tau = \frac{\gamma}{T} + R\frac{cT}{2} \quad (33)$$

$\tau$  will be minimized by choosing  $T$  to solve the first order condition:

$$-\frac{\gamma}{T^2} + R\frac{c}{2} = 0 \quad (34)$$

that is,

$$T = \sqrt{\frac{2\gamma}{Rc}}. \quad (35)$$

From (31) and (35), average real money balances will then be

$$\frac{\bar{m}}{P} = \frac{cT}{2} = \sqrt{\frac{c\gamma}{2R}}. \quad (36)$$

Barro observes that while this equation indicates economies of scale in holding real money balances (since real balances rise less than proportionately with real transactions  $c$ ), the cost of time, and thus the real costs of transactions  $\gamma$  will be positively correlated with  $c$ . This will tend to make average real money balances move more closely to proportional to  $c$ .

- More generally, we can write the demand for *average* real transactions balances as

$$\frac{\bar{m}}{P} = \phi(R, c, \gamma) \quad (37)$$

with  $R$  affecting  $\phi$  negatively,  $c$  affecting  $\phi$  positively and real transactions costs  $\gamma$  affecting  $\phi$  positively.

- If the conversion times are then distributed randomly across individuals, the economy-wide *per capita* real money balances will equal the average money balances held by each individual as

given in (37). Equivalently, we can write aggregate real money demand as

$$\frac{M}{P} = \Phi(R, c, \gamma). \quad (38)$$

### 7. *Money Neutrality*

- Suppose there is a once and for all change in the level of the money stock  $M$ . Since the remainder of the model (the goods and labor market clearing conditions and the constituent demand and supply functions) do not involve  $M$  or  $P$ , none of the real variables in (38) will be affected by a change in  $M$  or  $P$ . Hence, equilibrium will require a proportionate movement in  $P$  equal to the original movement in  $M$ . As with stationary states in the Sidrauski model, changes in the money supply will be neutral with a change in  $M$  merely resulting in an equi-proportionate change in nominal prices  $P$ .
- An increase in  $M$  holding  $P$  fixed will produce an incipient increase in household wealth. The positive wealth effect would tend to raise consumption demand and reduce labor supply. The incipient excess demand for goods would bid prices up until real money balances decline to their original level and the initiating wealth effect disappears.
- While increases in money supply will raise nominal prices, they are not the only source of price changes in the model. In particular, any other shocks that effect the equilibrium real interest rate, the level of real transactions, or real transactions costs  $\gamma$  will affect the demand for money and thus average nominal prices – even if the nominal quantity of money is unchanged.

### 8. *Inflation*

- Inflation is a continuing increase in average nominal prices. While real shocks can affect average nominal prices by altering the demand for money, these shocks are unlikely to produce a persistent rise in prices – particularly of any more than a negligible magnitude. For example, we could imagine continuing improvements in transactions technology producing modest declines in

money demand of one or two percent a year, with a corresponding low inflation rate. The only feasible source of substantial persistent inflation, however, is substantial persistent growth of money supply. This observation is borne out by both time series and cross-sectional data.

- A nominal interest rate  $R$  reflects a promise to pay a lender  $R$  *currency* units in the future for each unit of currency lent today. When there is persistent inflation, however, lenders will anticipate that tomorrow's currency will be worth less in real terms than today's currency. They will demand compensation not only for foregoing consumption for a period of time (this is the *real* interest rate) but also for the *expected* loss in the purchasing power of money. The *real* return on a bond with a nominal interest rate of  $R$  when the expected inflation rate is  $\pi^e$  will be

$$r = \frac{1 + R}{1 + \pi^e} - 1 = \frac{R - \pi^e}{1 + \pi^e} \quad (39)$$

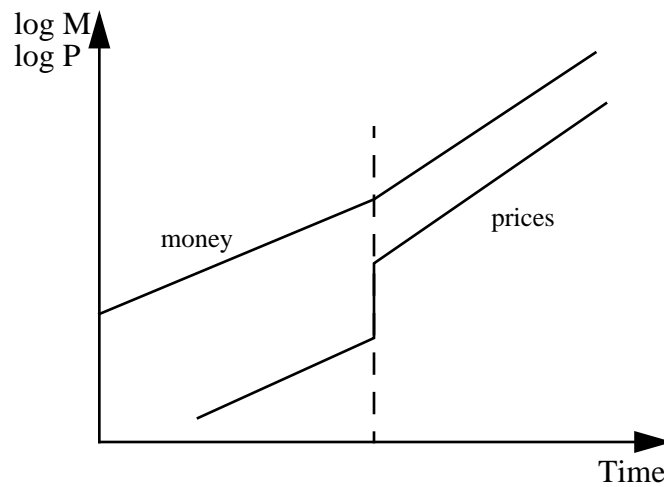
Alternatively, for a real interest rate of  $r$  and an expected inflation rate of  $\pi^e$ , the *nominal* interest rate will be

$$R = r + \pi^e + r\pi^e \quad (40)$$

Note that for both  $r$  and  $\pi^e$  small, the product  $r\pi^e$  will be very small and we can write the *expected* real interest rate (which is the variable that affects  $c$ ,  $i$ ,  $n^s$  and so on) as approximately  $R - \pi^e$ .

- A change in the rate of growth of the money supply will alter the inflation rate in long run equilibrium. Since this in turn will alter the nominal interest rate  $R$ , the return on holding money, and the demand for real money balances will be affected. Again, however, if  $R$  fully adjusts to the expected inflation rate, no real variables will be affected. The change in the demand for real money balances associated with a change in the inflation rate can be accommodated by an additional change in the price *level*.
- For example, suppose the money supply is growing constantly at the rate  $\mu$  and then the rate of

growth is increased to  $\mu'$  and maintained at that higher level. Assume individuals are fully aware of the increased money growth and immediately adjust their forecast of inflation accordingly. With an unchanged real interest rate  $r$ , the nominal interest rate will immediately rise to reflect the difference between  $\mu'$  and  $\mu$ . The demand for real money balances will decline and the average *level* of nominal prices will jump immediately the new higher money growth rate is instigated. The situation can be represented in the following diagram:



### 9. *Expectations and Hyperinflation*

- Cagan argued that in a hyper-inflation (which he defined as an inflation with prices rising more than 50% per month) by far the most dominant influence on the demand for money will be changes in the *expected* rate of inflation. While hyper-inflationary episodes will often be associated with changes in real income, wealth, and real rates of return on bonds and other assets, the effects of such changes on the demand for money will be small in comparison with changes in  $\pi^e$ .
- Cagan used the above idea to explain the drop in *real* money balances characteristic of hyper-inflations. It is typically observed in hyper-inflations that prices actually rise faster than the rate of money growth so that real balances fall. Cagan argued that this fall in real balances can be explained by the rise in the cost of holding money balances caused by higher *expected* rates of

inflation.

- Cagan hypothesized that the expected rate of inflation could be modeled as an exponentially-weighted distributed lag on past actual rates of inflation. Equivalently, he assumed expectations were *adapted*, or updated, to account for the error made in the previous period. Expressing his model in discrete time we get

$$\pi_{t+1}^e = \pi_t^e + \beta(\pi_t - \pi_t^e); \beta > 0 \quad (41)$$

- Note that the equation (41) for expected inflation can be written

$$\pi_{t+1}^e = (1 - \beta)\pi_t^e + \beta\pi_t \quad (42)$$

and using the lag operator we can re-write (42) as

$$\pi_{t+1}^e = \frac{\beta}{1 - (1 - \beta)L} \pi_t. \quad (43)$$

- We then conclude that if prices were constant at some period  $t = 0$  in the past and expectations for inflation in period  $t = 1$  were also for zero inflation, then the first order difference equation (41) has the solution:

$$\pi_t^e = \beta \sum_{\tau=1}^{t-1} (1 - \beta)^{t-1-\tau} \pi_\tau \quad (44)$$

Thus, the current expected inflation rate is an exponentially weighted average of past actual inflation rates.

- To focus on the effect of expected inflation on money demand Cagan postulated a demand for real money balances of the form

$$\ln[M_t/P_t] = -\alpha\pi_t^e - \gamma \quad (45)$$

We can also write the money demand function (45) as

$$\ln M_t = \ln P_t - \alpha \pi_t^e - \gamma \quad (46)$$

Now take the quasi-difference of (46) and use (42) to eliminate the terms in  $\pi_t^e$  to get

$$\ln M_{t+1} - (1-\beta)\ln M_t = \ln P_{t+1} - (1-\beta) \ln P_t - \alpha\beta \pi_t - \beta\gamma \quad (47)$$

or

$$\ln M_{t+1} = (1-\beta)\ln M_t + \ln P_{t+1} - (1-\beta) \ln P_t - \alpha\beta \pi_t - \beta\gamma \quad (48)$$

which is an equation solely in terms of the observable variables  $M$ ,  $P$  and  $\pi$ .

- Hence estimates of  $\beta$ ,  $\alpha$  and  $\gamma$  can be obtained.<sup>4</sup> Note that we in fact get *two* estimates of  $\beta$ . The parameter  $\beta$  is said to be *over-identified*.
- Cagan found that, with the above method of relating expectations to observed data on inflation rates, this equation performs very well in explaining the movement in real money balances. He notes that in several cases, however, that observations in months toward the end of the hyper-inflationary period do not fit the equation very well and he ascribes this to his expectational equation. If individuals come to believe that the hyper-inflation may be coming to an end, they may no longer form their expectations about future rates of inflation solely on the basis of past actual rates. He also notes that the different values of  $\beta$  he finds for different countries suggests that  $\beta$  is not constant but depends on the recent history of inflation experienced by individuals in the economy. In economies where rapid inflation has more recently been experienced inflationary expectations appear to respond more rapidly to innovations in the inflation rate in the sense that  $\beta$  is closer to 1.

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<sup>4</sup>This is not the way Cagan originally obtained the estimates - instead he formed  $\pi_t^e$  for different values of  $\beta$  and performed a grid search to determine which values of  $\beta$  worked best.

- Sargent (“The Ends of Four Big Inflations”) focused attention on the closing months of hyper-inflationary episodes. Sargent makes the point that as the inflation rate subsides at the end of the episode, the demand for real money balances will rise. As a result, large increases in the supply of money can be accommodated by increasing money demand without inflation having to rise. The ends of hyper-inflations are often characterized by falling inflation rates despite the fact that the supply of money continues to grow at a rapid pace. Sargent notes that in the episodes he studies the expansion of money at the end of the hyper-inflation only occurs as individuals exchange gold, foreign exchange or other assets for domestic money. In this way, the central bank can be sure that the increase in the domestic money supply is all demanded by the public and so will be non-inflationary.
- We are lead to ask why the hyper-inflation comes to an end if the money supply continues to grow at a rapid rate. Equivalently, why does money demand start to rise again? Why should individuals believe the hyper-inflation is over? Sargent argues, along with Cagan, that the money expansion is used primarily as a tax to finance government expenditure in excess of explicit tax receipts. Inflation is a tax on the holding of real money balances. The base of the tax is the level of real balances; the tax rate is the rate of depreciation in the real value of money, or the rate of rise in prices. Sargent and Cagan both note that if a government begins to finance its expenditure by simply printing more money and using the new issue to purchase goods and services, the subsequent rise in inflation will further worsen the government budget deficit. Explicit taxes are levied on income or consumption from previous years. With an increase in the inflation rate, the real value of tax collections falls and the government turns to ever-higher rates of inflation to finance its expenditure. Sargent argues that *believable fiscal reform* and a credible commitment to only expand the money supply in line with money demand are the necessary prerequisites to ending a hyper-inflation. Once individuals come to believe the government is no longer going to finance its expenditure with the inflation tax, the money supply can continue to expand for a time, without inflation being re-ignited, because of increases in money demand.