- In this chapter we look at two modifications of stochastic growth models to account for two features of business cycles. The first is a complication to investment to introduce more short run dynamics into the model. The second involves changing household preferences to allow for variable labor supply.

1. *Time to build investment*

- Kydland and Prescott (1982) introduced the idea that it takes time to build new productive capital structures. They assumed that household utility takes a modified Cobb-Douglas form. This can be motivated by appealing to the long run evidence that labor supply is relatively inelastic. Specifically, in the Cobb-Douglas form with "share parameter" θ, the fraction of hours supplied to the market tends to be close to θ and thus relatively constant over time.

- Kydland and Prescott generalize the Cobb-Douglas function, however, by allowing the curvature of the overall function to change

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\theta} z_t^{1-\theta})^{1-\gamma}}{1-\gamma} \tag{1}$$

  In (1), the parameter γ can be interpreted as the *intertemporal elasticity of substitution* for both consumption c and "leisure" (or non-market uses of time) z.

- Kydland and Prescott (1982) showed how, by allowing for "time to build" capital one can produce a simple model that has investment more volatile than consumption, has most aggregates moving pro-cyclically and allows productivity shocks to account for about 50% of post-1950 business cycle variability.

- Specifically, the productive capital stock k_t is assumed to evolve according to

$$k_{t+1} = (1-\delta)k_t + s_{1t} \tag{2}$$

where s_{jt}, j = 1,…,J is new capital that is j periods from completion. Thus s_{jt} evolves according to $s_{jt+1} = s_{j+1t}$. Each of the s_{jt} become additional state variables for the dynamic programming problem.

- Let the investment required at stage j of the process be φ_j per unit of capital. Then the total fixed investment at time t will be

$$\sum_{j=1}^{J} \varphi_j s_{jt} \tag{3}$$

- Kydland and Prescott also allow for inventory investment. If y_t is the inventory stock at the beginning of period t, then inventory investment during period t will be $y_{t+1} - y_t$. Total investment in period t will thus be

$$I_t = \sum_{j=1}^{J} \varphi_j s_{jt} + y_{t+1} - y_t \tag{4}$$

- Output is taken to be constant elasticity of substitution (CES) between current production and the beginning of period inventory

$$[(1-\psi)(\varepsilon_t k_t^{\alpha}(1-z_t)^{1-\alpha})^{-\nu} + \psi y_t^{-\nu}]^{-\frac{1}{\nu}} \tag{5}$$

where $1/(1+\nu)$ is the elasticity of substitution. Using (4) and (5), the aggregate resource constraint now becomes

$$c_t + \sum_{j=1}^{J} \varphi_j s_{jt} + y_{t+1} - y_t = [(1-\psi)(\varepsilon_t k_t^{\alpha}(1-z_t)^{1-\alpha})^{-\nu} + \psi y_t^{-\nu}]^{-\frac{1}{\nu}} \tag{6}$$

- Kydland claims that by adding the "time to build" technology and inventory, using "reasonable" parameter values from the microeconomic evidence, and identifying productivity shocks with the "Solow residual", the model can account for key properties of the US business cycle. It can match the relative volatility of investment and consumption, and the pro-cyclicality of most aggregates. The model can also account for about half the volatility in US output. The major problem with the model, however, is that it cannot explain employment variation. Kydland therefore turned to intertemporal substitution as a solution.

2. *Intertemporal Substitution in Labor Supply*

- To account for the fact that output variation over the business cycle is associated with a variation in the level of employment we need to drop the growth model assumption of an inelastic labor supply.

- Consider again the static problem

$$\max_{c,\,h} U(c, h) \quad \text{subject to } pc = wh \tag{7}$$

  The consumer chooses to allocate time to market or non-market activity based on the *real wage*. As we noted when we examined the simple model of labor supply, to be consistent with the long run evidence on changes in average hours worked or lifetime labor supply as real wages change, the utility function needs to produce a labor supply curve that is relatively inelastic. But this then makes it difficult to use the same model to account for significant employment variation over business cycles.

- We look to intertemporal substitution to provide a source of large *short-run* fluctuations. Casual observation, such as that holidays or work hours over the week can be shifted with little or no compensation required, suggests that the elasticity of the *distribution* of work supplied over time is high. Life cycle participation decisions, the timing of education and training, or investment in a new job via search can also be influenced by intertemporal wage movements.

- To illustrate further, consider again the two period model we examined at the beginning of the course

$$\max_{c_0, h_0, c_1, h_1} U(c_0, h_0, c_1, h_1) \text{ subject to } p_0c_0 + p_1c_1 \leq w_0h_0 + w_1h_1 \tag{8}$$

where $p_1 = \dfrac{p_1^*}{1 + r}$ and $w_1 = \dfrac{w_1^*}{1 + r}$.

- Using this model we want to find a U consistent with:

  (a) the evidence that labor supply is inelastic in the long run

  (b) the evidence that employment variation can be considerable in the short run.

- Solely to facilitate a graphical analysis, we make three simplifying assumptions:

  (i) Impose the assumption of a long run inelastic labor supply by assuming *total* labor supply over the two periods is fixed at $\bar{h} = h_0 + h_1$. This is equivalent to implicitly imposing restrictions on the utility function U.

  (ii) Assume utility is separable between consumption and labor supply. That is, suppose the utility obtained from c_1 versus c_2 is independent of the amount of work done in each period.

  (iii) Define y as real income in present value terms and assume all income is consumed in either of the two periods. Let $\theta_0 = c_0/y$ and $\theta_1 = c_1/y$ be the shares of consumption in the two periods. Define a price index P by

$$P = \theta_0 p_0 + \theta_1 p_1 \tag{9}$$

The intertemporal budget constraint can then be written in terms of P as

$$c_0p_0 + c_1p_1 = [\theta_0 p_0 + \theta_1 p_1]\, y = Py = w_0h_0 + w_1h_1 \tag{10}$$

- Now split the original maximization problem into two sub-problems. We can first consider the problem of maximizing utility by choosing consumption taking labor supplies in the periods as given. We can then maximize the overall level of utility by choosing the two levels of labor supply. We introduce y as a new choice variable along with a second budget constraint.

- Thus, consider the problem

$$\max_{c_0, c_1} U(c_0, h_0, c_1, h_1) \text{ subject to } p_0c_0 + p_1c_1 = Py \tag{11}$$

With utility separable between consumption and labor supply, we can solve this problem for consumption as a function of real prices in the two periods and real income, y:

$$c_0 = c_0\left(\frac{p_0}{P}, \frac{p_1}{P}, y\right) \text{ and } c_1 = c_1\left(\frac{p_0}{P}, \frac{p_1}{P}, y\right) \tag{12}$$

- Define the indirect utility function that incorporates the maximizing consumption decisions by

$$V\left(h_0, h_1, y; \frac{p_0}{P}, \frac{p_1}{P}\right) = U\left(c_0\left(\frac{p_0}{P}, \frac{p_1}{P}, y\right), h_0, c_1\left(\frac{p_0}{P}, \frac{p_1}{P}, y\right), h_1\right) \tag{13}$$

The labor supply decision can then be represented by the maximization problem

$$\max_{h_0, h_1, y} V\left(h_0, h_1, y; \frac{p_0}{P}, \frac{p_1}{P}\right) \text{ subject to the constraint } \frac{w_0}{P}h_0 + \frac{w_1}{P}h_1 = y \tag{14}$$

and the constraint that follows from the assumption that long run labor supply elasticity is zero:[1]

$$h_0 + h_1 = \bar{h} \tag{15}$$

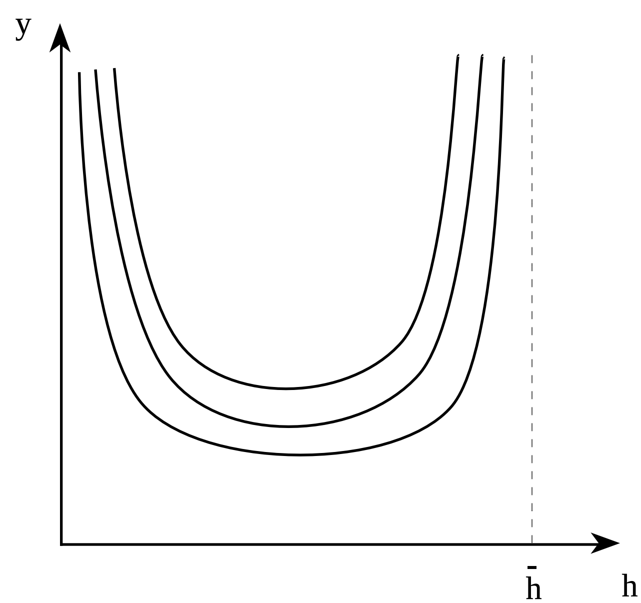
Define $h = h_0$ and put $\bar{h} - h = h_1$.

[1] The implicit restrictions on U involved in this assumption can be imposed more formally by setting the problem up as a three stage maximization. At the first stage, choose $\bar{h}$. Then choose h and y as outlined above, and finally choose the consumption levels c_0 and c_1. We then want U to be such that $\bar{h}$ is very inelastic with respect to a change in prices over the two periods that keeps real wages constant.

- To focus on intertemporal allocation, assume that real wages are constant over the two periods by letting $w_i = \omega p_i$. As an aside we should note that, although real wage variation might be important in some business cycle episodes, the empirical evidence does not show a consistent relationship between real wages and cyclical variations in employment. The labor supply maximization can then be written as:

$$\max_{h,\,y} V\left(h, \bar{h} - h, y; \frac{p_0}{P}, \frac{p_1}{P}\right) \text{ subject to } \frac{p_0}{P}h + \frac{p_1}{P}(\bar{h} - h) = \frac{y}{\omega} \tag{16}$$

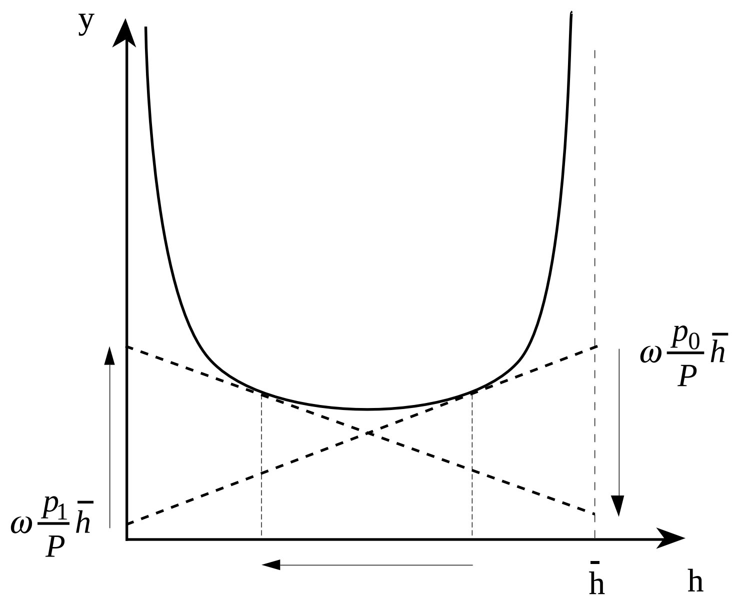
- Assuming individuals are fairly indifferent to the allocation of time across the two periods, indifference curves for the indirect utility function V in (h,y) space will be as illustrated in the following diagram. It indicates that individuals object to supplying almost all the time in one period



or the next but there is a wide range of distribution of work time across the two periods that leaves them more or less indifferent. Higher real income, y, of course makes individuals better off. Note that even though V is a function of the allocation of consumption across the two periods and so responds to changes in p_0/P and p_1/P, as long as individuals have access to capital markets and can borrow and lend at the same interest rate r in the first period, as we are assuming, we would expect this general pattern for the indifference curves to remain valid.

- The position of the budget line depends on p_0/P and p_1/P and thus future relative to current prices. In particular, a decrease in real interest rates r will tend to raise p_1 relative to p_0.

- We can represent the maximization as below. As the diagram shows, if we raise p_1 relative to p_0, current labor supply h decreases. Of course, in general the **same** indifference curve need not be tangent to the two budget lines - that is, the change in p_0 relative to p_1 is likely in general to change maximized utility. The *size* of the labor supply response depends, however, on the flatness of the indifference curves. If they are very flat over a broad range of values of h then a small change in p_0 relative to p_1 can produce a large change in labor supply in the current period h.



- As a result of intertemporal substitution, the short-run labor supply curve might be much more elastic (that is, the opportunity cost of not working is much less) than analyses based on long run labor supply would suggest. Further, fixed costs of employment might also imply a more elastic labor supply schedule for a "representative" individual than for any one individual.

- In particular, we conclude that, as a result of intertemporal substitution, current labor supply responds positively to the real interest rate. If substitution effects dominate wealth effects in the labor-leisure choice, we would also find that current labor supply responds positively to an increase in the real wage.

- In the two period model, we have taken utility of labor supply to be non-separable over time. The utility cost of working today depends on the amount of labor supplied in other periods. Such a formulation would appear to be difficult to incorporate into a dynamic programming framework.

3. *Intertemporal substitution in a dynamic programming model*

- Kydland and Prescott (1982) also showed how intertemporal substitution can be incorporated into a stochastic growth model. Specifically they modified household preferences by introducing another stock variable to the model. We can let current leisure contribute to a stock of "recent leisure" just like current investment contributes to the stock of physical capital. Furthermore, there can be a depreciation rate on this stock so that as periods of non-market time fade into the past they have less of an effect on current decisions. The desire to take leisure in any given period is then lessened if the current stock of past leisure recently enjoyed is higher.

- Specifically, the utility function is assumed to take the form

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\theta} z_t^{1-\theta})^{1-\gamma}}{1-\gamma} \tag{17}$$

but where now z_t is not current leisure or non-market work, but rather a stock variable

$$z_t = \lambda z_{t-1} + \mu \ell_t \tag{18}$$

and current non-market time is denoted ℓ_t.

- With this model, the dynamic programming problem has z_t as an additional state variable. When combined with the "time to build" and inventory features discussed above, the model accounted for additional features of the business cycle including pro-cyclical movements in employment.

- Benhabib, Rogerson and Wright (1991) further modified the utility function (17) to allow c_t to be a CES function between home-produced and marketed goods and services. Home production then is allowed to have its own productivity shock.

- Home production allows hours in the consumption sector to respond positively to productivity shocks. In addition to increasing hours to accumulate more capital while productivity is high, there is an incentive to substitute market for home production of consumption goods when market productivity is high. In fact, the addition of a household production sector could result in labor flowing from the household sector to all market sectors during upswings rather than a transfer of labor from the consumption goods to the investment goods sector.

- Another problem with the standard model that home production can alleviate is that it introduces another source of shocks to the system. The standard model yields a very high positive correlation between productivity and output or productivity and hours. This is inconsistent with the data and can be alleviated by introducing productivity shocks in the household sector that are less than perfectly correlated with market productivity shocks.

- However, the model does not include the time to build or inventory accumulation features of the earlier Kydland and Prescott model. While the Benhabib, Rogerson and Wright model can match some business cycle characteristics better than the Kydland and Prescott model it is less successful at matching others. Another problem with the household production model is that it is very difficult to get independent confirmation of reasonable values for the parameters governing home production. This makes it difficult to verify how well the model truly can account for business cycle facts.

4. Fixed Costs of Employment

- The intertemporal substitution model might seem to be more suited to explaining variations in hours worked over the business cycle than variations in numbers of employees. To explain the latter we need to introduce fixed costs of employment.

- An interesting attempt to account for *employment* fluctuations over the business cycle is the indivisible labor model of Gary Hansen (JME 1985). Hansen starts with household preferences of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1 - h_t) \tag{19}$$

where c_t is consumption in period t and h_t is hours worked in period t. However, he assumes that labor is supplied in indivisible units - either households work an exogenously fixed number of hours in period t or they do not work at all. This introduces a non-convexity into the household maximization problem.

- More generally, we would like to look at a model where individuals can substitute leisure across time (to account for variations in overtime and average weekly hours worked over the cycle) but where there are also fixed costs of employment. As a result of the latter, individuals have limitations on their ability to vary hours and to some extent face employment lotteries with relatively fixed hours of work but a variable probability of being employed. A completely general model is difficult to analyze because of non-convexities. Hansen and Rogerson make simplifying assumptions in order to make the objective function for the household concave. More recent papers have followed up this work by using more flexible, but also more complicated specifications, that again cannot be solved analytically but instead can only be simulated using “reasonable” parameter values.

- Hansen and Rogerson assume that households choose lotteries rather than hours worked. Specifically, each period, instead of choosing hours, households choose a probability α_t of working a fixed number h_0 of hours and a lottery then determines whether or not any particular household works. Since the contract is being traded, the household is paid whether or not it works. In effect, the firm is providing complete unemployment insurance to the workers.

- Since all households are identical, they all choose the same contract – that is, the same α_t. However, *ex-post* a fraction α_t of the households will work and $1-\alpha_t$ will be *ex-post involuntarily* unemployed. Per capita hours worked in period t will be given by $\alpha_t h_0$. Each household, however, faces an uncertain prospect of supplying hours to market activity and so chooses α to maximize *expected utility*. Hansen assumes that utility in each period is separable in consumption and leisure

$$U(c_t, 1-h_t) = u(c_t) + v(1-h_t) \tag{20}$$

Then consumption in each period will be independent of whether or not the household works. Expected utility will be given by

$$EU(c_t, 1-h_t) = \alpha_t [u(c_t)+v(1-h_0)] + (1-\alpha_t) [u(c_t)+v(1)] = A + u(c_t) + B\alpha_t \tag{21}$$

for constants A and B. In particular, we can treat the representative household as if it had preferences which were *linear* in the probability of working α_t. For maximizing utility, we can ignore the constant A in the per-period utility function.

- Since households are paid for the *expected* amount of time they spend working, the household budget constraint becomes

$$c_t + i_t \leq w_t \alpha_t h_0 + r_t k_t \tag{22}$$

instead of the straightforward generalization of the Cass-Koopmans model we might otherwise expect.

- Thus the problem solved by the typical household is

$$\max \sum_{t=0}^{\infty} \beta^t [u(c_t) + B\alpha_t] \text{ subject to } c_t + i_t \leq w_t \alpha_t h_0 + r_t k_t \text{ and } k_{t+1} = (1-\delta)k_t + i_t \tag{23}$$

with k_0 and h_0 given. The representative firm will choose labor input h_t (by varying the *number* of workers actually employed rather the hours each employee works) to equate the marginal product of labor to the real wage as in the competitive version of the simple Cass-Koopmans model. The firm also demands capital services from households to equate the marginal product of capital to the real rental rate r_t which the firm takes as given.

- Note that since $\alpha_t = h_t/h_0$ we can equivalently model the representative household as choosing h_t to maximize an intertemporal utility function which is linear in h_t. The elasticity of substitution between leisure in different periods for the *aggregate* economy is *infinite* and independent of the willingness of individuals to substitute leisure across time. Hansen shows that this model is able to more closely mimic features of aggregate time series data, in particular the response of hours worked to a change in productivity, than a similar model without indivisible labor.