## 1. A Stochastic Growth Model

- One attempt to understand business cycles involves modifying the basic growth models to incorporate random shocks. The source of these shocks could be, for example, technological progress in particular industries or shocks to the supply of the factors of production.

- To provide some understanding of how these stochastic growth models work, we first examine a stochastic version of the Cass-Koopmans model for particular utility and production functions. Later, we shall look at models which provide better approximations to actual business cycle behavior.

- Suppose consumer preferences are given by

$$E\left\{\sum_{t=0}^{\infty}\beta^{t}\ln(c_t)\right\} ;\ 0 < \beta < 1 \tag{1}$$

and technology by

$$k_{t+1} = \varepsilon_t k_t^{\alpha} - c_t ;\ 0 < \alpha < 1 \tag{2}$$

where $\varepsilon_t$ is a random variable (representing, for example, random technological progress, random fluctuations in factor supplies or weather fluctuations). We assume $\varepsilon_t$ takes positive values with $\ln(\varepsilon_t)$ independently identically distributed $N(0, \sigma^2)$. *After* $\varepsilon_t$ becomes known, output $\varepsilon_t k_t^{\alpha}$ is divided between consumption $c_t$ and capital accumulation $k_{t+1}$.

- We shall show that the consumption savings policy which maximizes (1) subject to (2) is

$$c_t = (1 - \alpha\beta)\varepsilon_t k_t^{\alpha} \tag{3}$$

$$k_{t+1} = \alpha\beta\, \varepsilon_t k_t^{\alpha} \tag{4}$$

## *2. The Bellman Principle and the Value Function*

- We first want to solve this problem using the standard methodology. Later we shall show how you can exploit special features of the current problem to obtain the solution more simply.[1]

- Specifically, let $V(k_0,\varepsilon_0)$ denote the maximized value of the objective function (1) subject to the constraints (2) given that, in time period 0, the capital stock is $k_0$ and the shock is $\varepsilon_0$. That is, let

$$V(k_0, \varepsilon_0) \equiv \max_{\{c_t\}, \{k_{t+1}\}} E_0\left\{\sum_{t=0}^{\infty} \beta^t \ln(c_t)\right\} \tag{5}$$

where the maximization is carried out subject to the set of constraints (2) and $k_0$ is the initial capital stock. Here $E_0$ denotes the expected value conditional on information known at time 0. In our problem, all shocks, capital stocks, and consumptions up to and including period 0 are known at time 0.

- Now note that the original problem looks the same in each time period except that the capital stock and current productivity shock are different. The capital stock and current productivity shock are “state variables” for the problem. The Bellman principle of optimality says

$$V(k_t, \varepsilon_t) \equiv \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta E_t V(k_{t+1}, \varepsilon_{t+1})\} \tag{6}$$

where the maximization is carried out subject to the constraint (2) for time period t. Thus, $V(k_{t+1},\varepsilon_{t+1})$ represents the maximized value of the objective from t+1 on, and if the path {c, k} is maximizing it must maximize over the two sub-intervals [t, t+1] and then [t+1, ∞].

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[1] A homework problem had you solve the deterministic problem exploiting the special properties of this example.

## 3. *Guessing a Candidate Value Function*

- The solution method now involves guessing a form for the function V, and verifying that the guess works by showing that the first order conditions are satisfied.[2] We shall show that V has the form

$$V(k_0,\varepsilon_0) = A_0 + A_1\ln(k_0) + A_2\ln(\varepsilon_0) \tag{7}$$

There is no need to justify this guess - either it will work, or we will find out it is wrong and hopefully how to modify it to make it right. The procedure is analogous to solving integrals in calculus. Effectively, you have to guess the answer and then show your guess is right by differentiating to see if you do indeed get back the original function.

## 4. *A Motivation for the Guess*

- While the solution procedure does not require us to motivate where the guess came from, it might be useful to some students to see a motivation. The following manipulations do, however, involve a considerable amount of algebra.

- In our case, we have already guessed, based on our previous analysis of the related deterministic growth problem, that the maximizing c and k sequences are given by (3) and (4). Obviously (3) and (4) satisfy the budget constraints (they wouldn't be good guesses for the maximizing choices if they didn't - this is a minimal check on their suitability). Also, if (3) represents the maximizing choices for $c_t$, the function V will be given by

$$V(k_0, \varepsilon_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln[(1-\alpha\beta)\varepsilon_t k_t^{\alpha}] \tag{8}$$

Now use

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[2] We should also check that the second order conditions are satisfied. The concavity of ln and $x^{\alpha}$ will ensure the extremum is a maximum and not a minimum in our case.

$$\ln[(1 - \alpha\beta)\varepsilon_t k_t^{\alpha}] = \ln(1 - \alpha\beta) + \ln\varepsilon_t + \alpha\ln k_t$$

to write our guess for V as:

$$V(k_0, \varepsilon_0) = E_0\left\{\frac{\ln(1-\alpha\beta)}{1-\beta} + \sum_{t=0}^{\infty}\beta^t\ln(\varepsilon_t) + \alpha\sum_{t=0}^{\infty}\beta^t\ln(k_t)\right\} \tag{9}$$

Now use (4). Take logs of (4) to obtain the first order stochastic difference equation

$$\ln(k_{t+1}) = \ln(\alpha\beta) + \ln(\varepsilon_t) + \alpha\ln(k_t).$$

Conclude that $\ln(k_{t+1})$, will, if our guessed solution is correct, have the distribution

$$\ln(k_t) = \ln(\alpha\beta)(1+\alpha+\alpha^2+...+\alpha^{t-1}) + \alpha^t\ln(k_0) + \ln(\varepsilon_{t-1}) + \alpha\ln(\varepsilon_{t-2}) + \dots + \alpha^{t-1}\ln(\varepsilon_0)$$

- Taking expectations at time 0, and using the fact that $\ln(\varepsilon)$ is independently distributed with mean 0

$$E_0 \ln(k_t) = \ln(\alpha\beta)(1+\alpha+\alpha^2+...+\alpha^{t-1}) + \alpha^t\ln(k_0) + \alpha^{t-1}\ln(\varepsilon_0);\ t \geq 1 \tag{10}$$

Substitute (10) into the expression (9) for V to get

$$V(k_0, \varepsilon_0) = \frac{\ln(1-\alpha\beta)}{1-\beta} + \alpha\ln(k_0) + \ln(\varepsilon_0) + \tag{11}$$

$$\alpha\sum_{t=1}^{\infty}\beta^t[\ln(\alpha\beta)(1 + \alpha + \alpha^2 + \dots + \alpha^{t-1}) + \alpha^t\ln(k_0) + \alpha^{t-1}\ln(\varepsilon_0)]$$

where we have again used the fact that $E_0$ of future $\ln(\varepsilon)$ are all 0. Thus, we make an “intelligent guess” for V which takes the form

$$V(k_0, \varepsilon_0) = \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha}{1-\alpha\beta}\ln(k_0) + \frac{1}{1-\alpha\beta}\ln(\varepsilon_0) \tag{12}$$

which we can write as (7).

## 5. *Showing the guess is correct*

- Substitute our guess (7) for V into the Bellman equation (6). We want to verify that the following equality is valid:

$$V(k_t, \varepsilon_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta E_t[A_0 + A_1 \ln(k_{t+1}) + A_2 \ln(\varepsilon_{t+1})]\} \tag{13}$$

where the maximization on the right hand side is subject to the budget constraint

$$k_{t+1} = \varepsilon_t k_t^{\alpha} - c_t \tag{14}$$

Now observe that the "state variables" $\varepsilon_t$ and $k_t$ and the time t choice variables $c_t$, $k_{t+1}$ are known at time t. In addition, $\ln(\varepsilon_t)$ is independently identically distributed so $E_t \ln(\varepsilon_{t+1})$ is 0. Thus, under our guess for V, the function to be maximized on the right hand side of (13) can be simplified to

$$\ln(c_t) + \beta A_0 + \beta A_1 \ln(k_{t+1}) \tag{15}$$

- Define the Lagrangian

$$L = \ln(c_t) + \beta A_0 + \beta A_1 \ln(k_{t+1}) + \lambda(\varepsilon_t k_t^{\alpha} - c_t - k_{t+1}) \tag{16}$$

The first order conditions for a maximum of L are

$$\frac{1}{c_t} = \lambda \tag{17}$$

$$\frac{\beta A_1}{k_{t+1}} = \lambda \tag{18}$$

$$\varepsilon_t k_t^{\alpha} = c_t + k_{t+1} \tag{19}$$

- Substitute (17) and (18) into (19) to find

$$\frac{1}{\lambda}(1+\beta A_1) = \varepsilon_t k_t^{\alpha} \tag{20}$$

Thus, if our guess for V is correct, maximizing $c_t$ and $k_{t+1}$ will be given by

$$c_t = \frac{1}{1+\beta A_1}\varepsilon_t k_t^{\alpha} \tag{21}$$

$$k_{t+1} = \frac{\beta A_1}{1+\beta A_1}\varepsilon_t k_t^{\alpha}. \tag{22}$$

- Now substitute these solutions for $c_t$ and $k_{t+1}$ back into the right hand side of Bellman's equation (13) and we ned to verify that

$$V(k_t, \varepsilon_t) = A_0 + A_1 \ln k_t + A_2 \ln \varepsilon_t = \ln\left(\frac{1}{1+\beta A_1}\varepsilon_t k_t^{\alpha}\right) + \beta\left[A_0 + A_1 \ln\left(\frac{\beta A_1}{1+\beta A_1}\varepsilon_t k_t^{\alpha}\right)\right] \tag{23}$$

is valid for any values of $k_t$ and $\varepsilon_t$. Expand the terms on the right side of (23) and write it as

$$\beta A_0 - (1+\beta A_1)\ln(1+\beta A_1) + \beta A_1 \ln \beta A_1 + \alpha(1+\beta A_1)\ln k_t + (1+\beta A_1)\ln \varepsilon_t \tag{24}$$

and conclude that the functional equation will indeed be valid if the constants $A_0$, $A_1$ and $A_2$ satisfy

$$\begin{aligned} A_0 &= \beta A_0 - (1+\beta A_1)\ln(1+\beta A_1) + \beta A_1 \ln \beta A_1 \\ A_1 &= \alpha(1+\beta A_1) \\ A_2 &= (1+\beta A_1) \end{aligned} \tag{25}$$

that is, the guess will be correct if we choose the constants to be:

$$
\begin{aligned}
A_1 &= \frac{\alpha}{1-\alpha\beta} \\
A_2 &= \frac{1}{1-\alpha\beta} \\
A_0(1-\beta)(1-\alpha\beta) &= \ln(1-\alpha\beta) + \alpha\beta\ln\left[\frac{\alpha\beta}{1-\alpha\beta}\right]
\end{aligned}
\tag{26}
$$

- For the correct value of $A_1$ given in (26), the maximizing $c_t$ will be as originally guessed in (3) and the maximizing $k_{t+1}$ will be as originally guessed in (4). This can be verified by substituting the value for $A_1$ in (26) into (21) and (22).

## *6. Alternative Approach Specific to this Problem*

- A more direct approach to this particular problem is possible. You can transform the problem and consider choosing a sequence of savings rates:

$$
x_{t+1} = \frac{k_{t+1}}{\varepsilon_t k_t^{\alpha}} \tag{27}
$$

Using the constraints (2), rewrite (1) in terms of $x_t$ and maximize it directly.[3] This is much easier than going through the valuation function V but it only works for this problem. However, this alternative method was probably the way the "guesses" were originally obtained. In fact, it is often difficult to arrive at guesses for solutions to the functional equation. It is possible to numerically approximate the solution for V for more general functional forms. I discuss the numerical approximation of V when I teach the dynamic optimization course.

## *7. Implications of the Model*

- We want to see what the model implies about the cyclical behavior of output, consumption, factor prices and so on. We can then compare the implications with the empirical evidence.

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[3] Do this yourselves for homework!

- If we were to associate the optimal planning solution with a competitive equilibrium, the equilibrium returns to capital ($r_t$) will be its marginal product

$$r_t = \alpha \varepsilon_t k_t^{\alpha - 1} \tag{28}$$

while the return to the fixed factor of production ($w_t$) will be

$$w_t = y_t - r_t k_t = (1 - \alpha)\varepsilon_t k_t^{\alpha}. \tag{29}$$

- Given $k_0$, we can use (4) to find the distribution for $\ln k_t$:

$$\ln k_{t+1} = \ln \alpha\beta + \alpha \ln k_t + \ln \varepsilon_t \tag{30}$$

Hence,

$$\ln k_1 = \ln \alpha\beta + \alpha \ln k_0 + \ln \varepsilon_1 \tag{31}$$

$$\ln k_2 = (1+\alpha)\ln \alpha\beta + \alpha^2 \ln k_0 + \ln \varepsilon_2 + \alpha \ln \varepsilon_1 \tag{32}$$

and generally

$$\ln k_t = (1+\alpha + \ldots + \alpha^{t-1})\ln \alpha\beta + \alpha^t \ln k_0 + \ln \varepsilon_t + \alpha \ln \varepsilon_{t-1} + \ldots + \alpha^{t-1} \ln \varepsilon_1 \tag{33}$$

- By assumption, $\ln(\varepsilon_t)$ is iid $N(0,\sigma^2)$, so $\ln k_t$ will also be iid normal and, since $\alpha < 1$, $\ln k_t$ will have a stationary limiting distribution that also is normal with mean $\mu_k = E \ln k_t = E \ln k_{t+1}$ satisfying

$$\mu_k = \ln \alpha\beta + \alpha\mu_k + 0; \text{ that is } \mu_k = \frac{\ln \alpha\beta}{1 - \alpha} \tag{34}$$

and a variance satisfying

$$E[\ln k_{t+1} - \mu]^2 = \alpha^2 E[\ln k_t - \mu]^2 + E[\ln \varepsilon_t]^2 \tag{35}$$

since $\ln k_t$ and $\ln \varepsilon_t$ are uncorrelated. That is,

$$\sigma_k^2 = \frac{\sigma^2}{1 - \alpha^2} \tag{36}$$

- Now real GNP $y_t = \varepsilon_t k_t^{\alpha}$, so $\ln y_t = \alpha \ln k_t + \ln \varepsilon_t$ and

$$(1 - \alpha L) \ln y_t = \alpha(1 - \alpha L) \ln k_t + (1 - \alpha L) \ln \varepsilon_t$$

$$= \alpha(\ln \alpha\beta + \ln \varepsilon_{t-1}) + (1 - \alpha L) \ln \varepsilon_t = \alpha \ln \alpha\beta + \ln \varepsilon_t \tag{37}$$

From this we can deduce that

$$\mu_y = \frac{\alpha \ln \alpha\beta}{1 - \alpha} \tag{38}$$

$$\sigma_y^2 = \frac{\sigma^2}{1 - \alpha^2} \tag{39}$$

$$\mathrm{cov}[\ln y_t, \ln y_{t-k}] = \alpha^k \sigma_y^2 \tag{40}$$

- Next we want to find the limiting values of $\mathrm{cov}[\ln(y_t), \ln(c_t)]$, $\mathrm{cov}[\ln(y_t), \ln(r_t)]$ and $\mathrm{cov}[\ln(y_t), \ln(w_t)]$. First we need to find the stochastic difference equation $\ln c_t$, $\ln r_t$ and $\ln w_t$ satisfy. From (28),

$$\ln r_t = \ln \alpha + (\alpha - 1) \ln k_t + \ln \varepsilon_t \tag{41}$$

so that

$$(1 - \alpha L) \ln r_t = (1 - \alpha) \ln \alpha + (\alpha - 1)(\ln \alpha\beta + \ln \varepsilon_{t-1}) + (1 - \alpha L) \ln \varepsilon_t$$

$$= (\alpha - 1) \ln \beta + \ln \varepsilon_t - \ln \varepsilon_{t-1} \tag{42}$$

that is,

$$\ln r_t = -\ln \beta + \ln \varepsilon_t + (\alpha - 1)\,[\ln \varepsilon_{t-1} + \alpha \ln \varepsilon_{t-2} + \ldots\,] \tag{43}$$

which implies

$$\mu_r = -\ln\beta \tag{44}$$

$$\sigma_r^2 = \frac{2\sigma^2}{1+\alpha} \tag{45}$$

$$\begin{aligned} \operatorname{cov}[\ln y_t, \ln r_t] &= E\left[\frac{\ln\varepsilon_t}{1-\alpha L}\left(\frac{\ln\varepsilon_t}{1-\alpha L} - \frac{\ln\varepsilon_{t-1}}{1-\alpha L}\right)\right] \\ &= \sigma^2[(1+\alpha^2+\ldots) - \alpha(1+\alpha^2+\ldots)] = \frac{1-\alpha}{1-\alpha^2}\sigma^2 = \frac{\sigma^2}{1+\alpha} \end{aligned} \tag{46}$$

Similarly, from (29), $\ln w_t = \ln(1-\alpha) + \alpha \ln k_t + \ln \varepsilon_t$, so that

$$\begin{aligned} (1-\alpha L)\ln w_t &= (1-\alpha)\ln(1-\alpha) + \alpha(\ln \alpha\beta + \ln \varepsilon_{t-1}) + (1-\alpha L)\ln \varepsilon_t \\ &= (1-\alpha)\ln(1-\alpha) + \alpha \ln \alpha\beta + \ln \varepsilon_t \end{aligned} \tag{47}$$

and hence

$$\mu_w = \frac{(1-\alpha)\ln(1-\alpha) + \alpha\ln\alpha\beta}{1-\alpha} \tag{48}$$

$$\sigma_w^2 = \frac{\sigma^2}{1-\alpha^2} \tag{49}$$

$$\operatorname{cov}[\ln y_t, \ln w_t] = \frac{\sigma^2}{1-\alpha^2} \tag{50}$$

Finally, from (3), $\ln c_t = \ln(1-\alpha\beta) + \alpha \ln k_t + \ln \varepsilon_t$, so that

$$(1-\alpha L)\ln c_t = (1-\alpha)\ln(1-\alpha\beta) + \alpha(\ln\alpha\beta + \ln\varepsilon_{t-1}) + (1-\alpha L)\ln\varepsilon_t$$

$$= (1-\alpha)\ln(1-\alpha\beta) + \alpha\ln\alpha\beta + \ln\varepsilon_t \tag{51}$$

and hence

$$\mu_c = \frac{(1-\alpha)\ln(1-\alpha\beta) + \alpha\ln\alpha\beta}{1-\alpha} \tag{52}$$

$$\sigma_c^2 = \frac{\sigma^2}{1-\alpha^2} \tag{53}$$

$$\operatorname{cov}[\ln y_t, \ln c_t] = \frac{\sigma^2}{1-\alpha^2} \tag{54}$$

- Comparing these predictions of the model to the empirical evidence on business cycle fluctuations, we find that simply adding shocks to a simple growth model will not result in a good model of the business cycle. Output growth does not cycle in this model, while consumption and payments to labor (inelastically supplied) are *perfectly* correlated with output growth. Further, the model implies that the variance of interest rates could be higher than the variance of output growth, consumption and labor income. This will be so if $\alpha < 0.5$.

## *8. Intertemporal and Intersectoral Correlations*

- There are a number of ways to address the serial correlation problem with the simple stochastic growth model. We want the “cycles” produced by our model to have an autocorrelation pattern matching the pattern observed in measured economic series. Specifically, we want the model to match the observed tendency for deviations from trend to persist for several years and then be reversed by a period of deviations of the opposite sign.

- The stochastic version of the simple growth model only gives a one-period lag as a result of the capital accumulation equation which makes $k_{t+1}$ depend on $k_t$. Kydland and Prescott introduced the notion that it takes time to install capital ("time to build") to explain lags in the evolution of output in response to exogenous shocks. This makes the current $k_t$ and therefore current output, depend on lagged $k_t$ several periods into the past. The result is a stochastic difference equation for output that has maximum lag length equal to the longest gestation period an investment project.

- To model *business cycles*, however, a mechanism is required to achieve a correlation between the random movements of output in *different* industries. We would also like the relative amplitudes of fluctuations in different sectors to correspond to our observations based on the evidence. A problem with the simple stochastic growth model is that it is hard to see why productivity shocks would be positively correlated across all industries at once. It seems plausible that productivity shocks could be a source of large fluctuations in the output of many industries, but why would we expect all sectors to experience positive or negative productivity shocks at the same time?

- Perhaps the simplest way to modify a standard growth model to account for correlation *between* output movements in different sectors of the economy, as observed in a "typical" business cycle, is to allow for there to be several industries which can use as inputs the outputs of each of the other industries. This can lead to correlated movements in the outputs of each of the industries even though a particular shock to aggregate output might originate in one particular industry. It can also introduce some dynamics into the model if outputs from one sector this period are used as inputs in another sector the following period. The maximum lag length in output then depends on the number of inter-related sectors in the economy.

- To illustrate this point, we shall discuss the model of Long and Plosser (JPE 1983). Long and Plosser allow for correlation between output fluctuations in different industries by noting that outputs of some industries are used as inputs in other industries. We shall show that allowing for these intersectoral influences also introduces autocorrelation into output fluctuations. As noted above, investment adjustment costs and delivery lags are an alternative way of explaining auto-correlation in output fluctuations.

### *9. Long and Plosser model*

- Long and Plosser postulate a representative consumer with preferences:

$$\sum_{t=0}^{\infty} \beta^t U(c_t, z_t); \; 0 < \beta < 1 \tag{55}$$

where $\beta$ is a discount factor, $c_t$ is an N×1 vector of commodity consumption in period t and $z_t$ is the amount of “leisure” (or other “non-market work”) time consumed in period t.

- The N commodities in the economy can be produced with a constant returns to scale production function

$$\begin{bmatrix} y_{1t+1} \\ y_{2t+1} \\ \vdots \\ \vdots \\ \vdots \\ y_{Nt+1} \end{bmatrix} = \begin{bmatrix} F_1(L_{1t}, X_{1t}; \lambda_{1t+1}) \\ F_2(L_{2t}, X_{2t}; \lambda_{2t+1}) \\ \vdots \\ \vdots \\ \vdots \\ F_N(L_{Nt}, X_{Nt}; \lambda_{1N+1}) \end{bmatrix} \tag{56}$$

where $y_{it+1}$ is the total stock of commodity i available at t+1

$F_i$ is concave and homogeneous of degree 1

$L_{it}$ is labor allocated to industry i at time t

$X_{it}$ is a 1×N vector of inputs (from each of the N industries) used in the production of commodity i at time t

$\lambda_{it+1}$ is a random shock to the output of commodity i at time t

- Consumptions $c_{it}$, intermediate inputs $X_{it}$ and outputs $y_{it}$ available at the beginning of period t satisfy the constraints

$$c_{it} + \sum_{j=1}^{N} X_{jit} = y_{it} \tag{57}$$

- The representative consumer has available a total time of H hours each period which can be allocated to leisure or producing each of the N commodities:

$$z_t + \sum_{i=1}^{N} L_{it} = H \tag{58}$$

- To show that this model can lead to random outputs from each industry and the economy as a whole which resemble business cycles, Long and Plosser examine a special case. They let utility be:

$$U(c_t, z_t) = \phi_0 \ln z_t + \sum_{i=1}^{N} \phi_i \ln c_{it} \tag{59}$$

where $\phi_i \geq 0$ for each i. They also take the production functions for each sector i to be:

$$y_{it+1} = \lambda_{it+1} L_{it}^{b_i} \prod_{j=1}^{N} X_{ijt}^{a_{ij}} \tag{60}$$

with the parameters $a_{ij}$ and $b_i$ non-negative and constant over time and, to give homogeneity of degree 1, $b_i + \Sigma_j a_{ij} = 1$.

- Since $\lambda_{t+1}$ is assumed to depend only upon $\lambda_t$ and not any values of $\lambda$ from periods previous to t, the current *state* of the economy, or the vector of variables which are sufficient to describe the choices open to the representative consumer, can be written $s_t = (y_t, \lambda_t)$.

- Let $V(s_t)$ be the value function for the representative consumer or the maximized value of his expected future utility given the current state of the economy. So we have

$$V(s_t) = \max_{c, L, X} E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} U(c_s, z_s)\right] \tag{61}$$

with the maximization carried out subject to the constraints (56), (57) and (58) above. The notation $E_t$ means the expected value is taken with the information about the future values of random variables being that available to the consumer at time t.

- As above, we can use the idea that if the consumer is maximizing from t to ∞ then the maximizing path must also be maximizing from t to t+1 and then from t+1 to ∞. Hence,

$$V(s_t) = \max \{U(c_t, z_t) + \beta\ E_t\ [V(s_{t+1})]\}. \tag{62}$$

- To find the solution to the consumer's problem we guess that V(.) takes the form:

$$V(s_t) = \textstyle\sum_i \gamma_i \ln y_{it} + J(\lambda_t) + K \tag{63}$$

with

$$\gamma_i = \phi_i + \beta \textstyle\sum_j \gamma_j a_{ji},\ i = 1,2,3,\ldots N \tag{64}$$

and

$$J(\lambda_t) = \beta E\{\textstyle\sum_i \gamma_i \ln\lambda_{it+1} + E_t\ [J(\lambda_{t+1})]\} \tag{65}$$

and K a constant which depends on the parameters in the utility and production functions but not on the elements of the state vector, $y_t$ or $\lambda_t$. Substitute this guess for V(.) into the right hand side of the functional equation (62) and we get

$$V(s_t) = \max \{U(c_t, z_t) + \beta\ E_t\ [\textstyle\sum_i \gamma_i \ln y_{it+1} + J(\lambda_{t+1}) + K]\} \tag{66}$$

But from (60)

$$\ln y_{it+1} = \ln\lambda_{it+1} + b_i \ln L_{it} + \sum_j a_{ij} \ln X_{ijt} \tag{67}$$

Also,

$$U(c_t, z_t) = \phi_0 \ln(H - \sum_i L_{it}) + \sum_i \phi_i \ln(y_{it} - \sum_j X_{jit}) \tag{68}$$

- Now look at the first order conditions for maximizing the right hand side of (66) with respect to the choice of inputs $L_{it}$ and $X_{jit}$:

$$-\phi_0/z_t + \beta\gamma_i b_i/L_{it} = 0, \forall\ i$$

$$-\phi_i/c_{it} + \beta\gamma_j a_{ji}/X_{jit} = 0, \forall\ i,j \tag{69}$$

From the first set of these conditions and the constraint (58) we obtain:

$$\tilde{z}_t = \frac{\phi_0 H}{\phi_0 + \beta\sum_i \gamma_i b_i} \tag{70}$$

$$\tilde{L}_{it} = \frac{\beta\gamma_i b_i H}{\phi_0 + \beta\sum_i \gamma_i b_i} \tag{71}$$

From the second set of the conditions in (69) we obtain:

$$\tilde{X}_{jit} = \frac{\beta\gamma_j a_{ji}}{\phi_i}\tilde{c}_{it} \tag{72}$$

Substitute this into the constraint (57) to obtain:

$$\left(\phi_i + \beta\sum_j \gamma_j a_{ji}\right)\tilde{c}_{it} = \phi_i y_{it} \tag{73}$$

or, using the definition of $\gamma_i$,

$$\tilde{c}_{it} = \frac{\phi_i}{\gamma_i} y_{it} \tag{74}$$

But then the solution of the first order condition for $\tilde{X}_{jit}$ implies that:

$$\tilde{X}_{jit} = (\beta\gamma_j a_{ji}/\gamma_i) y_{it} \tag{75}$$

- These solutions for the optimal values of the endogenous variables may be substituted into the right-hand side of the functional equation (62) to obtain:

$$\begin{gathered} V(s_t) = \phi_0 \ln[\phi_0 H/(\phi_0 + \beta\Sigma_i\gamma_i b_i)] + \Sigma_i\ \phi_i \ln[(\phi_i/\gamma_i)y_{it}] + \beta\ E_t\,[\Sigma_i\ \gamma_i\{\ln\lambda_{it+1} + \\ b_i \ln(\beta\gamma_i b_i H/(\phi_0 + \beta\Sigma_i\gamma_i b_i)) + \Sigma_j\ a_{ij} \ln((\beta\gamma_i a_{ij}/\gamma_j)y_{jt})\} + J(\lambda_{t+1}) + K] \end{gathered} \tag{76}$$

which is of the required form since we may write it:

$$\begin{gathered} V(s_t) = \beta K + \Sigma_i(\phi_i + \beta\Sigma_j\gamma_i a_{ij})\ln y_{it} + \beta E\{\Sigma_i\gamma_i \ln\lambda_{it+1} + E_t[J(\lambda_{t+1})]\} + \\ \Big\{ \phi_0 \ln[\phi_0\, H/(\phi_0 + \beta\Sigma_i\gamma_i b_i)] + \Sigma_i\phi_i \ln[\phi_i/\gamma_i] + \beta\Sigma_i\gamma_i b_i \ln(\beta\gamma_i b_i H/(\phi_0 + \beta\Sigma_i\gamma_i b_i)) + \\ \beta\Sigma_i\gamma_i\Sigma_j\ a_{ij} \ln(\beta\gamma_i a_{ij}/\gamma_j) \Big\} \end{gathered} \tag{77}$$

To have V(.) of the guessed form we define the constant K as the expression within the large braces divided by $(1 - \beta)$.

- Analogously to our discussion of the simple growth models, we can relate this “optimal planning problem” to a competitive equilibrium by defining the utility-denominated prices of commodities as the marginal utility values of increasing the supply of commodities available at time t and the utility-denominated wage rate as the utility value of a marginal increase in leisure consumption.

- Returning to the solutions for the optimal c's, z, X's and L's above we note the following about the predictions of the model:
  - the allocation of the available stock of a commodity is an increasing function of its value in that use
  - the amounts of a commodity allocated to each of its uses is an increasing function of its total supply
  - if the output of some commodity i is unusually high at time t then inputs of commodity i in all its uses will also be high at time t. This propagates the shock forward in time and spreads the future effects of the shock across sectors of the economy.
  - the allocation of any given commodity does not depend on the contemporaneously available amounts of other commodities
  - given $y_t$ none of the allocations made at time t depends on $\lambda_t$. In particular, the labor/leisure decision is independent of both $y_t$ and $\lambda_t$. It is argued that this is a consequence of the particular utility and production function chosen for the closed-form solution and not a necessary consequence of more general models of this type.
  - the relative price of commodity i is higher the greater its scarcity relative to other commodities
  - the greater the productivity of a commodity $a_{ij}$ the higher its relative price
  - the higher the preference for a commodity $\phi_i$ the higher its relative price
  - the greater the preference for leisure the higher the utility-denominated wage. The utility-denominated wage is also higher when the productivity of labor $b_i$ is higher.

- The most interesting feature of the model is its prediction of the evolution of outputs over time. If we substitute the optimal allocations of inputs $L_{it}$ and $X_{ijt}$ into the production function we get:

$$\ln y_{it+1} = \ln\lambda_{it+1} + b_i \ln[\beta\gamma_i b_i H/(\phi_0+\beta\textstyle\sum_i\gamma_i b_i)] + \textstyle\sum_j a_{ij}\ln[(\beta\gamma_i a_{ij}/\gamma_j)y_{jt}] \qquad (78)$$

or

$$\ln y_{it+1} = k + \sum_{i} a_{ij} \ln y_{jt} + \xi_{it+1} \tag{79}$$

which is a set of simultaneous first order stochastic difference equations. The vector of constants k will influence the short run movements of outputs as well as their steady-state values. In fact the solution to the stochastic difference equation can be written:

$$\ln y_{t+1} = (I - A)^{-1}k + \xi_{t+1} + A\xi_t + A^2\xi_{t-1} + \ldots \tag{80}$$

- The steady-state value of y is $(I - A)^{-1}k$. The input-output matrix $A = [a_{ij}]$ will have a crucial influence on the evolution of outputs. If $\xi$ is an i.i.d. process, the matrix A will completely summarize the propagation mechanism for the random shocks $\xi$. To get significant interaction between the variables, the A matrix should display lots of large off-diagonal elements. Because of the form of the optimal decision rules for consumption and commodity inputs to production, their behavior will mimic the behavior of the output vector.

- Note that if we let $(I - AL)^{-1} = (I - AL)^*/\det(I - AL)$ where $M^*$ is the transposed matrix of cofactors of matrix M and detM is its determinant, then the deviations of $\ln y_{t+1}$ from its long run mean value $(I-A)^{-1}k$ can be written $(I-AL)^*\xi_{t+1}/\det(I-AL)$ or if we denote by $\hat{y}$ deviations of lny from its mean we have:

$$\det(I - AL)\hat{y}_{t+1} = (I - AL)^*\xi_{t+1} \tag{81}$$

  Each of the components of $\hat{y}_t$ will have an autoregressive component of the same order, and since this order will in general be greater than two, the series can easily display a cyclical pattern of autocovariances (the autocovariances will solve a deterministic difference equation of order equal to the order of the polynomial in L $\det(I - AL)$).

- Long and Plosser examine a numerical solution for an input-output matrix similar to an aggregated version of the US input-output cost shares in 1967. They get agriculture and mining having a higher variance than manufacturing, which is not what one would expect for US data. In addition, mining and agriculture shocks do not appear to be highly correlated with business cycle fluctuations in the remainder of the economy.

- Their model also seems to imply a relatively much more variable services output than we found in the data. The main candidates for sources of business cycles in their model appear to be shocks either to manufacturing or services output. Only these sectors have enough *interaction* with all the remaining sectors to produce the widespread correlated response of output across sectors typical of the business cycle.

- Another problem with this model is that it does not give an adequate explanation of the much greater cyclical movement of investment and producer and consumer durables output than services or nondurables consumption. These problems might be addressed the same way we shall address them in a single sector model. Nevertheless, the model does show that the fact that industries are linked by the use of intermediate goods, and that maximizing consumers and firms will have an incentive to spread out the effects of shocks, can account for a considerable amount of the correlation and persistence amongst the movements of economic variables. Employment fluctuations might be explained by the models discussed in the next set of notes.