CASS-KOOPMANS MODEL AS A COMPETITIVE EQUILIBRIUM

1. *The Fundamental Welfare Theorems in a Dynamic Setting*

- In the second chapter, we discussed the relationship between competitive equilibria and Pareto optima in a simple two period model. In this chapter we illustrate how the argument can be extended to a dynamic framework by showing how the Cass-Koopmans model can be thought of as a competitive equilibrium.

- We shall restrict ourselves to the simplest model with no population growth. As in the previous chapter we retain the assumption that labor is inelastically supplied.

2. *A Competitive Equilibrium*

- We join to the optimum sequences $\{c_0, c_1, c_2, \ldots\}$ and $\{k_1, k_2, k_3, \ldots.\}$ two sets of futures prices, real wages $\{w_0, w_1, w_2, \ldots\}$, and real rental rates on capital $\{u_0, u_1, u_2, \ldots.\}$, such that when firms maximize profits taking prices as given, and consumers maximize utility also taking prices as given, the market-clearing quantities determined are the optimal sequences $\{c_0, c_1, c_2, \ldots\}$ and $\{k_1, k_2, k_3, \ldots\}$.

- More specifically, a set of prices will yield an equilibrium if the capital that households *choose* to supply to firms at those prices matches the demand for capital *chosen* by firms at the same prices (capital market equilibrium), the labor (inelastically) supplied by households matches the demand for labor *chosen* by firms at the given prices (labor market equilibrium), and the output supplied by firms matches the demand for output for consumption and investment from households (goods market equilibrium).

- An alternative assumption to the futures markets assumption is that we have a sequence of spot markets for n_t and k_t determining w_t and u_t. However, since k_t lasts indefinitely into the future, consumers cannot make a decision about c_t and k_t today without knowing all future prices, so we must effectively assume they have *perfect foresight* about future spot prices.

3. *The Representative Firm Maximization Problem*

- For each period t, the firms will solve the maximization problem:

$$\max_{n_t,\, k_t} F(k_t, n_t) - w_t n_t - u_t k_t \tag{1}$$

However, we imagine the firms solving all these problems simultaneously at time 0 (by dealing in futures markets).

- We allow the firms to choose labor demand at t, but *in equilibrium* prices must be such that the firms choose the labor that will be inelastically supplied by households in the per capita quantity n each period.

4. *The Representative Consumer Maximization Problem*

- The consumers are assumed to rent capital to firms at the *real* rental rates $\{u_0, u_1, u_2, \ldots\}$ and sell labor to the firms at the *real* wage rates $\{w_0, w_1, w_2, \ldots\}$ also determined in futures market trading at time 0.

- The consumers solve the maximization problem:

$$\max_{c_t,\, k_t} \sum_{t=0} \beta^t U(c_t) \tag{2}$$

subject to the constraints

$$c_t + k_{t+1} - (1-\delta)k_t \leq w_t n_t + u_t k_t;\ t = 0, 1, 2, \ldots \tag{3}$$

$$n_t = n \text{ for all t.} \tag{4}$$

- Observe that the household budget constraints can be re-written as

$$c_t + k_{t+1} \leq w_t n_t + (1+u_t-\delta)k_t \quad t = 0, 1, 2, \ldots \tag{5}$$

so that we can identify $u_t - \delta \equiv R_t$ as a *rate of return* on savings k_t that the household takes as given. The household chooses consumption c_t and savings k_{t+1} to maximize utility subject to a budget constraint that equates consumption and savings to the sum of labor and capital income.

5. A Guess for the Equilibrium Prices

- An equilibrium in this economy is a set of pricing *functions*. In order to solve such a problem, we have to guess the solution and then verify that our guess works. This is analogous to the way we solve other functional equations, such as differential equations, or the way we integrate.

- Suppose $\{\bar{c}_0, \bar{c}_1, \bar{c}_2,\ldots\}$ and $\{\bar{k}_1, \bar{k}_2, \bar{k}_3,\ldots\}$ solve the optimum (planning) problem. We guess:

$$\bar{u}_t = F_k(\bar{k}_t, n) \tag{6}$$

$$\bar{w}_t = F_n(\bar{k}_t, n) \tag{7}$$

will be the equilibrium prices to solve the competitive equilibrium.

6. Optimization by Representative Agents in Response to these Prices

- The prices (6) and (7) lead the profit-maximizing firm to choose the inputs $\{\bar{k}_t\}$ and $\{n\}$ in each period t (why?).

- The Lagrangian for the consumer will be:

$$L = \sum\{\beta^t U(c_t) + \beta^t q_t[\bar{w}_t n + \bar{u}_t k_t - c_t - k_{t+1} + (1-\delta)k_t]\} \tag{8}$$

with first order conditions for a maximum (apart from the transversality condition):

$$U'(c_t) = q_t \tag{9}$$

$$\beta^t q_t \bar{u}_t + \beta^t q_t(1-\delta) - \beta^{t-1} q_{t-1} = 0 \tag{10}$$

$$c_t + k_{t+1} = \bar{w}_t n + \bar{u}_t k_t + (1-\delta)k_t \qquad (11)$$

- Substitute the above guesses for the equilibrium prices into the first order conditions (10) and (11):

$$\beta^t q_t F_k(\bar{k}_t, n) + \beta^t q_t(1-\delta) - \beta^{t-1} q_{t-1} = 0 \qquad (12)$$

$$c_t + k_{t+1} = F_n(\bar{k}_t, n)n + F_k(\bar{k}_t, n)k_t + (1-\delta)k_t \qquad (13)$$

7. *Showing the Prices Result in an Equilibrium Allocation*

- We want to show the equations (12) and (13) describing the maximizing behavior by households, along with our guesses for prices (6) and (7), are solved by the same $\{c_t\}$ and $\{k_t\}$ sequences that solve the optimum (planning) problem.

- Note that if we put $c_t = \bar{c}_t$ and $k_t = \bar{k}_t$ in (13) then by Euler's theorem we get:

$$\bar{c}_t + \bar{k}_{t+1} = F(\bar{k}_t, n) + (1-\delta)\bar{k}_t \qquad (14)$$

and from (9)

$$U'(\bar{c}_t) = q_t \qquad (15)$$

- Now it is easy to see that (15), (12) and (14) must hold for $\{\bar{c}_t\}$ and $\{\bar{k}_t\}$ solutions to the first order necessary conditions (apart from the transversality condition) to the planning problem

$$L = \sum \{\beta^t U(c_t) + \beta^t q_t [F(k_t, n) + (1-\delta)k_t - c_t - k_{t+1}]\} \qquad (16)$$

That is, by the *definition* of $\{\bar{c}_t\}$ and $\{\bar{k}_t\}$ as the solutions to the planning problem we know they solve the equations:

$$U'(\bar{c}_t) = \bar{q}_t \tag{17}$$

$$\beta^t \bar{q}_t F_k(\bar{k}_t, n) + \beta^t \bar{q}_t (1 - \delta) - \beta^{t-1} \bar{q}_{t-1} = 0 \tag{18}$$

$$\bar{c}_t + \bar{k}_{t+1} = F(\bar{k}_t, n) + (1 - \delta)\bar{k}_t \tag{19}$$

- It is somewhat more difficult to prove the two problems lead to the same transversality condition and we shall not pursue that issue further in these lectures (see an article by Brock and Mirman on the reading list).

8. *A Trap to Avoid*

- It might be thought that we could merely substitute the Euler equation

$$F(\bar{k}_t, n) = F_n(\bar{k}_t, n)n + F_k(\bar{k}_t, n)\bar{k}_t \tag{20}$$

into the Lagrangian for the competitive consumer's maximization problem to give us the same Lagrangian as in the planning problem. However, this is not a valid argument. It amounts to assuming each consumer chooses the *equilibrium* capital stock whereas the idea behind a competitive equilibrium is that consumers and firms maximize taking prices as given.

9. *The Intertemporal Arbitrage Condition*

- Note that if we use the rate of return on savings defined above then the first order conditions for household maximization (9) and (10) can be written as

$$\beta\, U'(c_t)(1+R_t) = U'(c_{t-1}) \tag{21}$$

This equation can be interpreted as an *arbitrage* condition. If the household foregoes one unit of consumption at time t-1, the cost in utility terms is $U'(c_{t-1})$. If that unit of output were saved in the form of k_t it would yield $(1+R_t)$ units of output in period t. Each of these units of output have value $U'(c_t)$ at date t, but in terms of units of t-1 utility each of those time t units are only valued at fraction β.