

1. *An Economic Model of Aggregate Savings*

- We now introduce preferences to determine the proportion of income saved and invested and hence the growth of the capital stock. Why are we interested in modelling the choice of the savings rate as based on preferences? There are several reasons:

(i) We want to model market economies where the savings are determined by individual choices. Although we are interested in market economies, we look at the Pareto optimal problem first and then apply the fundamental welfare theorems.

(ii) Ultimately, we are interested in using our model to make predictions about the effects of various exogenous shocks. If the variation in an exogenous variable we are considering is within the range of historical experience,¹ we could examine the statistical evidence to see what the effect of the shock was in the past. However, often we are interested in using the model to make *out-of-sample* predictions. Unless the model is based on some structural theory we cannot have any confidence that the various parameters of the model will remain constant in a new economic environment. The basic paradigm of an *economic* theory is that we try to explain choice behavior as the response of a maximizing agent with a relatively stable objective function to changes in the agent's opportunities. If we allowed the objectives of agents to arbitrarily change we can “explain” or “rationalize” virtually any choice in an ex-post sense, but the model would be of no use for making predictions about the consequences of changes in exogenous variables.

(iii) Another reason for basing our aggregate theory on a model of maximizing agents is that we might be able to relate our model to other economic evidence and in this way gain greater confidence in the accuracy of our basic assumptions. In the present case, for example, we could look at household saving behavior as recorded in cross-sectional data sets and see if the same micro-economic assumptions on household preferences that enable us to explain the cross-sectional evidence also enable us to explain the aggregate time series evidence. We shall return to this point

¹And assuming also that the effect of the exogenous variable of interest can be identified in the data separately from the effects of other exogenous variables which happened to be changing at the same time

again when we discuss the permanent income hypothesis of consumption.

(iv) We observed last lecture that higher savings helped increase future prosperity, but higher current savings also reduced current consumption. This naturally raised the question of what is the “best” savings rate. So long as we take a “democratic” approach to answering such questions, the criterion for measuring the “value” of alternative savings rates must depend on individual preferences for consumption today versus consumption tomorrow.

2. *Saving by a Representative Consumer*

- The way we allow preferences to determine the saving rate s_t at time t is by introducing a *representative* consumer with an *intertemporal* utility function:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \quad (1)$$

with the discount factor β , $0 < \beta < 1$. We can write $\beta = 1/(1+\rho)$ and interpret ρ as the *rate of time preference* and think of it as a “natural rate of interest”.

- A characteristic of the above preferences is that they are *additively separable* over time. In other words, we imagine our representative consumer choosing a bundle of consumption at time period t , and then re-allocating this consumption across goods and services in each period in response to relative prices. The trade-off of consumption today for consumption tomorrow can be considered independently of the trade-off of consuming different goods within the time period. We know that for explaining microeconomic data on certain types of consumption this is likely to be a poor approximation. In particular, it is likely to work better for explaining consumption of non-durables than the consumption of durables. However, accounting for non-separability over time raises additional technical complications. Also, to avoid corner solutions where $c_t = 0$, we assume $U'(0) = \infty$.
- We retain the same technology we considered last lecture and we also retain the inelastic labor

supply assumption. Keeping the same model of population growth we can derive the same difference equation except that the savings rate is now indexed by t:

$$X_{t+1} = [s_t/(1+\lambda)] f(X_t) + [(1-\delta)/(1+\lambda)] X_t \quad (2)$$

where again we assume $f(0) = 0$, $\lim_{X \rightarrow 0} f'(X) = \infty$, $f'(\cdot) > 0$, $f''(\cdot) < 0$.

- It is more convenient in this model to work in terms of c_t rather than s_t . In each period,

$$s_t f(X_t) = f(X_t) - c_t \quad (3)$$

so the difference equation incorporating our assumptions about technology and labor force growth becomes:

$$X_{t+1} = [f(X_t) - c_t]/(1+\lambda) + [(1-\delta)/(1+\lambda)] X_t. \quad (4)$$

3. *The Maximization Problem*

- The maximization problem for the consumer is then to choose *sequences* c_0, c_1, c_2, \dots and X_1, X_2, X_3, \dots to maximize the objective function (1) subject to the *set* of constraints (4).²
- Define a set of Lagrange multipliers $\phi_t = \beta^t q_t$ and write the Lagrangian:

$$L = \sum_{t=0}^T \beta^t \{ U(c_t) + q_t [f(X_t) - c_t + (1-\delta)X_t - (1+\lambda)X_{t+1}] \} \quad (5)$$

Differentiate L with respect to the c_t 's, X_t 's and q_t 's to get the necessary conditions for a maximum:

$$\beta^t \{ U'(c_t) - q_t \} = 0; t = 0, 1, 2, \dots, T \quad (6)$$

²We allow $c_t > f(X_t)$ so we don't have to be concerned about any corners for c_t . We also should constrain $X_t \geq 0$ for each t, but we ignore this in the formal analysis for the time being.

$$\beta^t q_t [f'(X_t) + (1-\delta)] - \beta^{t-1} q_{t-1} (1+\lambda) = 0; t = 1, 2, \dots, T \quad (7)$$

$$X_{t+1} = [f(X_t) - c_t]/(1+\lambda) + [(1-\delta)/(1+\lambda)] X_t; t = 0, 1, 2, \dots, T \quad (8)$$

$$\beta^T q_T X_{T+1} = 0 \quad (9)$$

$$X_0 \text{ given} \quad (10)$$

Equation (9) comes from considering $\beta^T q_T X_{T+1} > 0$. If that were the case, the consumer would be better off consuming more in the final period and leaving consumption in all earlier periods the same. Hence, $\beta^T q_T X_{T+1} = 0$ is a necessary condition for optimality.

4. *Solving the First Order Conditions*

- From (6) we can invert U' to obtain $c_t = g(q_t)$. Substitute this into the budget constraint (8) to obtain

$$X_{t+1} = [f(X_t) - g(q_t)]/(1+\lambda) + [(1-\delta)/(1+\lambda)] X_t \quad (11)$$

From (7) we obtain:

$$\beta q_t [f'(X_t) + (1-\delta)] = q_{t-1} (1+\lambda) \quad (12)$$

We have reduced the first order conditions to a *pair* of simultaneous non-linear difference equations in (X_t, q_t) with the pair of “initial” conditions $X = X_0$ at $t = 0$ and $\beta^T q_T X_{T+1} = 0$.

5. *An Economic Interpretation of the Solution*

- We can interpret these first order conditions as follows:
 - (i) q is the shadow price of capital (or shadow value of changes in X_t) in current value terms - β^t converts it to present value. Thus, $U'(c_t) = q_t$ implies that the consumer should equate the marginal value of eating a unit of output now to the marginal value of investing it for the future.

(ii) From (7) we get

$$f'(X_t) = (1+\lambda)[q_{t-1}/q_t]/\beta - (1-\delta) \quad (13)$$

The right hand side of (13) has the dimensions of an interest rate. Writing it in terms of the rate of time preference ρ and supposing we are in a “stationary environment” where $q_{t-1} = q_t$,

$$f'(X_t) = \rho + \lambda + \rho\lambda + \delta \quad (14)$$

If q_t is changing there are also capital gains and losses to take into account. If q_t is rising the interest rate is effectively reduced and vice versa.

6. Investigating Dynamics with a Phase Diagram

- To explain the qualitative nature of the solution we use a phase diagram. Let $T \rightarrow \infty$ to get a *pair* of difference equations with one initial condition X_0 . But we also have the second condition:

$$\lim_{T \rightarrow \infty} \beta^T q_T X_{T+1} = 0 \quad (15)$$

as a necessary condition for optimality. Equation (15) is the so-called transversality condition at ∞ (TVC $_{\infty}$) and provides the second initial condition to give an optimum.

- We have a particular interest in *stationary points* of the pair of difference equations. X_t stationary at \bar{X} and q_t stationary at \bar{q} yields the two non-linear equations in \bar{X} and \bar{q} :

$$\bar{X} = \frac{f(\bar{X}) - g(\bar{q})}{1 + \lambda} + \frac{1 - \delta}{1 + \lambda} \bar{X} \quad (16)$$

$$\beta[f'(\bar{X}) + (1 - \delta)] = 1 + \lambda \quad (17)$$

As above re-write (17) as:

$$f'(\bar{X}) = \delta + \lambda + \rho + \rho\lambda \quad (18)$$

which is an equation independent of \bar{q} . Let the solution of this be

$$\bar{X} = \tilde{X}. \quad (19)$$

From (16) we get:

$$g(\bar{q}) = f(\bar{X}) - (\delta + \lambda)\bar{X} \quad (20)$$

which can be solved for $\bar{q} = \tilde{q}$ at $\bar{X} = \tilde{X}$.

- Also observe that

$$\bar{q} = U'(g(\bar{q})) = U'[f(\bar{X}) - (\delta + \lambda)\bar{X}] \quad (21)$$

Let X^* be the solution to $f(X^*) = (\delta + \lambda)X^*$. Then as $\bar{X} \rightarrow 0$ or $\bar{X} \rightarrow X^*$, $\bar{q} \rightarrow \infty$. Further,

$$\frac{d\bar{q}}{d\bar{X}} = U''[f(\bar{X}) - (\delta + \lambda)\bar{X}][f'(\bar{X}) - (\delta + \lambda)]. \quad (22)$$

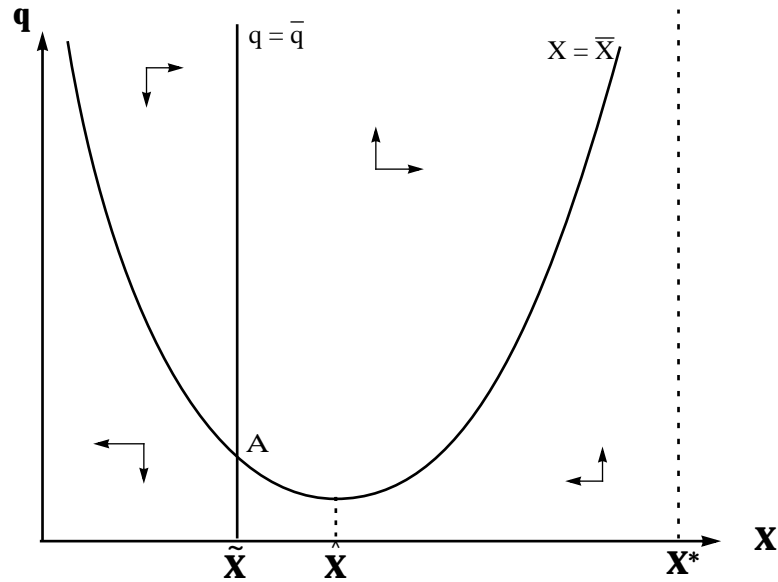
Let \hat{X} be the solution to $f'(\hat{X}) = \delta + \lambda$. (This is the so-called “golden rule” we discussed earlier.)

Then

$$\frac{d\bar{q}}{d\bar{X}} \begin{cases} < 0 \text{ for } 0 < \bar{X} < \hat{X} \\ > 0 \text{ for } \hat{X} < \bar{X} < X^* \end{cases} \quad (23)$$

- Finally, recall that we let \tilde{X} be the solution to $f'(\tilde{X}) = \delta + \lambda + \rho + \rho\lambda > \delta + \lambda$. Then $f'' < 0 \Rightarrow \tilde{X} < \hat{X}$. (Thus the “golden rule” results in a value of X which is too high.)

- Putting these results together we get the diagram:



Point **A** is the stationary point of the pair of difference equations.

- Now return to the original difference equations to investigate the dynamics. We have:

$$f'(X_t) = [q_{t-1}/q_t](1 + \lambda + \rho + \rho\lambda) + \delta - 1 \quad (24)$$

If $X > \tilde{X}$, $f'(X_t) < \delta + \lambda + \rho + \rho\lambda$ so that

$$[q_{t-1}/q_t](1 + \lambda + \rho + \rho\lambda) < (1 + \lambda + \rho + \rho\lambda) \quad (25)$$

which implies $q_{t-1} < q_t$, or that q is increasing.

- Also $g(q_t) = f(X_t) + (1-\delta)X_t - (1+\lambda)X_{t+1}$ so that:

$$q_t = U'[f(X_t) + (1-\delta)X_t - (1+\lambda)X_{t+1}]. \quad (26)$$

Then if $q_t > U'[f(X_t) - (\lambda + \delta)X_t]$

$$U'[f(X_t) + (1-\delta)X_t - (1+\lambda)X_{t+1}] > U'[f(X_t) - (\lambda + \delta)X_t] \quad (27)$$

and since $U'' < 0$

$$f(X_t) + (1 - \delta)X_t - (1 + \lambda)X_{t+1} < f(X_t) - (\lambda + \delta)X_t \quad (28)$$

which implies $X_t < X_{t+1}$ or X is increasing.

- There is a unique saddle path such that starting from any given initial X_0 , \exists a unique q_0 on the saddle path so that $(q_t, X_t) \rightarrow (\tilde{q}, \tilde{X})$. Mathematically, q_0 has to be chosen so that the TVC_∞ holds. The requirement that the terminal value of the capital stock $\rightarrow 0$ as $t \rightarrow \infty$ will determine a unique initial value of the capital stock consistent with maximization of utility. Note that we also have implicit constraints in the above problem that $X_t \geq 0$, $q_t \geq 0$ for all t . We did not impose these explicitly to avoid discussion of corners and Kuhn-Tucker conditions. Some of the non-optimal paths are ruled out because they violate these non-negativity constraints. Others are ruled out because they do not yield

$$\lim_{T \rightarrow \infty} \beta^T q_T X_{T+1} = 0 \quad (29)$$

7. *Choice of Initial Conditions*

- What do we make of the implication that if q_0 is not chosen correctly the economy might “go down a bad path”?
 - (i) Mathematically, the model is internally consistent. The TVC_∞ delivers a unique path. We have set up the mathematics to deliver us a unique optimal solution to the maximization problem.
 - (ii) This is the “planning model”, not on the surface at least a model of a decentralized economy. Perhaps taking explicit account of feedback mechanisms can lower the stakes in choosing the right q_0 at the start. However, we shall see that the same uniqueness problem arises in a competitive equilibrium.
 - (iii) The need to get the price of capital correct seems to be a potential source of instability in a market economy. Large corrections in the price of capital would appear to provide the potential

for exogenous shocks to the economy.

(iv) This model is, however, not suited to examine such issues. The aim of a model is to deliver predictions - hopefully testable ones. If the model says “anything can happen” then the model is saying nothing - there can be no possibility of finding recalcitrant data. Thus, if we are going to have shifts from one path to another as a feature of our description of a market economy, we need to model what causes these shifts.

8. *Bellman’s equation*

- In the analysis above, we obtained the first order conditions for maximizing the utility of the representative consumer by defining Lagrange multipliers and setting up a constrained optimization problem with an infinite number of constraints. Technically, we don’t have a theorem that enables us to write a constrained optimization problem as a Lagrangian when there are an infinite number of constraints.
- We now outline an alternative approach that can be justified more rigorously and which also can be used when there are shocks. The alternative approach uses the “Bellman principle of optimality”. This says that in order that the total sum of payoffs over the time interval $[0, \infty]$ be maximized the subtotal of the sum of payoffs over $[0,1]$ and $[1, \infty]$ must be maximized.³
- Observe that once we know X_τ in any period τ , the maximization problem is completely specified. This follows from the infinite planning horizon and the budget constraints. Preferences over consumption from remaining periods after τ are given by

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau} U(c_t) \tag{30}$$

³For the Bellman principle to hold, the problem needs to be such that the planned choices in the future as a function of possible future information match the actual choices when the future arrives. This is a property known as *time consistency*. The time separable utility function ensures our problem is time consistent.

while the budget constraints

$$X_{t+1} = [f(X_t) - c_t]/(1+\lambda) + [(1-\delta)/(1+\lambda)] X_t$$

depend only on the value of X_τ and the sequence of choice variables c_t . Define the *value function* $V(X)$ as the value of the objective function (30) for the economy with $X_\tau = X$ if the planner behaves optimally from period τ on.

- We do not discuss the conditions under which V can be shown to exist, to be differentiable and to be unique, but the conditions are satisfied by the current model. Some are discussed in the Appendix (chapter 20) of the Ljungqvist and Sargent text, and the issue will be discussed in more detail in the Dynamic Optimization course (ECON 523).
- Assuming that V is well-defined, the Bellman principle of optimality implies

$$V(X_t) = \max_{c_t, X_{t+1}} \{U(c_t) + \beta V(X_{t+1})\} \quad (31)$$

where the maximization is carried out subject to the constraint

$$X_{t+1} \leq \frac{f(X_t) - c_t + (1 - \delta)X_t}{1 + \lambda} \quad (32)$$

- Since the constraint will hold with equality if $U' > 0$, we can use (32) to eliminate one of the choice variables in (31). The maximization problem becomes

$$V(X_t) = \max_{c_t} \left\{ U(c_t) + \beta V \left[\frac{f(X_t) - c_t + (1 - \delta)X_t}{1 + \lambda} \right] \right\} \quad (33)$$

- Equation (33) is a special case of the equation (2.5) discussed in Ljungqvist and Sargent.
- The first order condition for the choice of c_t can be written

$$(1 + \lambda)U'(c_t) - \beta V' \left[\frac{f(X_t) - c_t + (1 - \delta)X_t}{1 + \lambda} \right] = 0 \quad (34)$$

- Equation (34) gives the optimal policy choice c_t as a function of the current state of the system X_t . To solve (34), however, we need to know the derivative of the unknown function V . The Bellman equation approach replaces the problem of solving for an infinite sequence of control variables by solving a functional equation for an unknown function.
- A change in X_t will change the optimal level of c_t , but again we use the envelope theorem. Since c solves (34), the derivative of V with respect to c_t is zero. The derivative of V with respect to X_t thus is simply the *partial* derivative:

$$V'(X_t) = \beta V'(X_{t+1}) \left\{ \frac{f'(X_t) + 1 - \delta}{1 + \lambda} \right\} \quad (35)$$

- Paralleling the development above, define the variable

$$q_t = U'(c_t) \quad (36)$$

Then we can again invert U' to obtain $c = g(q)$ and, substituting for c into the budget constraint, we obtain

$$X' = [f(X) - g(q)]/(1+\lambda) + [(1-\delta)/(1+\lambda)] X \quad (37)$$

- Now observe that the first order condition (34) for the choice of c can be written

$$(1 + \lambda)q_t = V'(X_{t+1}) \quad (38)$$

and assuming that c_{t-1} was also chosen optimally we therefore can lag (38) on period to obtain

$$V'(X_t) = (1 + \lambda)q_{t-1} \quad (39)$$

- Substituting (38) and (39) into the envelope result (35) we then obtain the second difference equation

$$(1 + \lambda)q_{t-1} = \beta q_t \{f'(X_t) + 1 - \delta\} \quad (40)$$

- Equation (40) is also known as the *Euler equation* for this model (see Ljungqvist and Sargent p35). It can also be written as a second order difference equation in X alone. Using the definition of q_t and the budget constraint (32) (as an equality), (40) implies

$$(1 + \lambda)U'[f(X_{t-1}) + (1 - \delta)X_{t-1} - (1 + \lambda)X_t] = \beta [f'(X_t) + 1 - \delta] U'[f(X_t) + (1 - \delta)X_t - (1 + \lambda)X_{t+1}] \quad (41)$$

- To ensure a unique solution, we need two initial conditions for (41). One is the value of X at $t=0$. The other is the transversality condition (TVC). By contrast, for “standard” U and f functions, the Bellman equation will have a unique solution. In a sense, the TVC arises because we “lose” a constant term when we convert from the value function to the derivative form (35).

9. *Continuous Time Optimal Control*⁴

- The original development of this model by Cass actually sets the model up in continuous time and uses optimal control theory to reach similar conclusions to those above. We shall not discuss optimal control theory in any detail in this course, but a few remarks are in order. The material covered here is further explored in the Dynamic Optimization class (ECON 523).
- The basic optimal control problem can be stated as follows. We want to choose a *path* (continuous function of time) for the **control** variable $u(t)$ to maximize the value of some integral objective function:

⁴These notes are derived from: “Economic Dynamics: An Optimal Control Framework” by W. A. Brock. The notes in this chapter are not essential to the remainder of the course, which focuses on discrete time models.

$$\int_{t_0}^T v(x, u, s) ds + B(x(T), T) \quad (42)$$

where $v(\dots)$ is the **instantaneous** payoff function, x is the **state** variable, t_0 is the current period and T the time horizon and $B(\dots)$ is the **bequest** function giving the value of $x(T)$ remaining at time T . The maximization is carried out subject to some **differential** constraint equation which relates the change in the state variable to the current value of the state and control variables:

$$\dot{x} = f(x, u, t) \quad (43)$$

The value of the state variable is also constrained at time t_0 :

$$x(t_0) = y. \quad (44)$$

10. *The Hamiltonian and the corresponding first order conditions*

- The “cookbook” way of solving this problem involves defining a **Hamiltonian** function (somewhat analogous to a Lagrangian) by introducing a variable p called the **costate** variable (analogous to a Lagrange multiplier):

$$H(p, x, u, t) = v(x, u, t) + p(t)f(x, u, t). \quad (45)$$

The necessary conditions for the control path $u(t)$ to be optimal then are, first, the co-state equations:

$$\dot{p} = -H_x^* \quad (46)$$

$$\dot{x} = H_p^* \quad (47)$$

$$x(t_0) = y \quad (48)$$

where H^* is the maximized value of H . Also, if we define,

$$V(y, t_0) \equiv \max_{u(t)} \int_{t_0}^T v(x, u, s) ds + B(x(T), T) \quad (49)$$

subject to

$$\dot{x} = f(x, u, t) \text{ and } x(t_0) = y, \quad (50)$$

we must have the Hamiltonian-Jacobi-Bellman equation satisfied:

$$V_t = -\max_{u(t)} H = -H^* \quad (51)$$

with

$$p = V_x \quad (52)$$

The transversality conditions:

$$V(x(T), T) = B(x(T), T) \text{ and } p(T) = B_x(x(T), T)$$

must also be satisfied.

11. An Intuitive Explanation

- The basic idea behind these conditions is as follows. We first use the “Bellman principle of optimality” to observe that in order that the total sum of payoffs from $x(t_0) = y$ over $[t_0, T]$ be maximized the subtotal of the sum of payoffs from $x(t_0+h)$ over $[t_0+h, T]$ must be maximized:

$$\begin{aligned}
V(y, t_0) &= \max \left\{ \int_{t_0}^{t_0+h} v(x, u, s) ds + \int_{t_0+h}^T v(x, u, s) ds + B(x(T), T) \right\} \\
&= \max \left\{ \int_{t_0}^{t_0+h} v(x, u, s) ds + \max \left[\int_{t_0+h}^T v(x, u, s) ds + B(x(T), T) \right] \right\} \\
&= \max \left\{ \int_{t_0}^{t_0+h} v(x, u, s) ds + V(x(t_0+h), t_0+h) \right\}
\end{aligned} \tag{53}$$

- Now use the mean value theorem of integral calculus to write:

$$\int_{t_0}^{t_0+h} v(x, u, s) ds = v(y, u, t_0)h + o(h) \tag{54}$$

where $o(h)$ is a function of order h that satisfies

$$\lim_{h \rightarrow 0} \frac{o(h)}{h} = 0 .$$

- Also expand $V(x(t_0+h), t_0+h)$ in a Taylor series about $x(t_0) = y$ and t_0 :

$$V(x(t_0+h), t_0+h) = V(y, t_0) + V_x(y, t_0) \Delta x + V_t(y, t_0) h + o(h) \tag{55}$$

and note that $\Delta x = x(t_0+h) - x(t_0) = f(y, u, t_0) h + o(h)$.

- Putting these results together, we get

$$V(y, t_0) = \max \{ v(y, u, t_0)h + V(y, t_0) + V_x(y, t_0)f(y, u, t_0)h + V_t(y, t_0)h + o(h) \} \tag{56}$$

Subtract $V(y, t_0)$ from each side of this equation, divide by h and let h tend to zero to get

$$0 = \max \{ v(y, u, t_0) + p f(y, u, t_0) + V_t(y, t_0) \} \tag{57}$$

where we have defined the variable $p = V_x(y, t_0)$. Rearrange the equation to find

$$-V_t = \max_{u(t)} H \quad (58)$$

The principle that the optimal control must maximize the Hamiltonian is called the **maximum principle**. It says the control should be chosen to maximize the sum of current instantaneous payoff $v(y, u, t_0)$ and future instantaneous value

$$p\dot{x} = pf(y, u, t_0) \quad (59)$$

where p is the shadow value of x , $p = V_x(y, t_0)$.

- To obtain the co-state equations, note that $\dot{x} = f(x, u, t)$. However, we can recover this equation via an alternative argument. If we differentiate H^* with respect to p and use the envelope theorem to drop terms involving changes in the endogenous variable u we get

$$H_p^* = f(x, u, t) = \dot{x} \quad (60)$$

Next differentiate

$$-V_t = H^* \quad (61)$$

with respect to x to get

$$-V_{tx} = H_x^* \quad (62)$$

But $V_{tx} = d/dt[V_x] = \dot{p}$.

- The transversality conditions require that at T the value of the problem equal the bequest value and the marginal value of extra x equal the marginal bequest value of additional x . If there is an inequality constraint $x(t) \geq 0$ for all t , but $B \equiv 0$ then the transversality condition $p(T) =$

$B_x(x(T), T)$ takes the form $p(T)x(T) = 0$; which says that nothing of value is left at terminal date T . When T is infinite, for a large class of problems the condition takes the form

$$\lim_{T \rightarrow \infty} p(T)x(T) = 0$$

and is called the **transversality condition at ∞** .

- When the objective function involves a constant rate of time discounting and the state transition equation does not depend directly on time the above first order conditions simplify. Specifically, let the objective be

$$\int_{t_0}^T e^{-\rho s} v(x, u) ds + e^{-\rho T} B(x(T)) \quad (63)$$

and the state transition equation be $\dot{x} = f(x, u)$, again with initial condition $x(t_0) = y$.

Introduce a change of units by defining

$$W(y, t) = e^{\rho t} V(y, t), \quad q = e^{\rho t} p, \quad \text{and } h = e^{\rho t} H \quad (64)$$

and the optimality conditions can be written:

$$\rho W - W_t = \max_u h(q, x, u) = h^*(q, x) \quad (65)$$

$$h(q, x, u) = v(x, u) + qf(x, u) \quad (66)$$

$$\dot{q} = \rho q - h_x^*, \quad \dot{x} = h_q^*, \quad x(t_0) = y \quad (67)$$

$$W(x(T), T) = B(x(T)) \quad \text{and} \quad q(T) = B_x(x(T)) \quad (68)$$

We also have that since $p = V_x$, $q = W_x$.

- When the horizon $T \rightarrow \infty$, W becomes independent of T so that $W_t = 0$; and the transversality condition becomes

$$\lim_{t \rightarrow \infty} e^{-\rho t} q(t)x(t) = 0 \quad (69)$$

12. *The Cass Koopmans Growth Model in Continuous Time*

- Apply the above analysis to our simple one sector growth model (without population growth or depreciation of capital). We have the instantaneous payoff function

$$v = e^{-\rho t} U(c(t)) \quad (70)$$

and the differential state transition equation

$$\dot{k}(t) = f(k(t)) - c(t). \quad (71)$$

The Hamiltonian becomes $U(c) + q[f(k) - c]$ so that the co-state equations can be written

$$\dot{q} = \rho q - q f'(k) \quad (72)$$

$$\dot{k} = f(k) - c \quad (73)$$

Also, we have that the control c must be chosen to maximize the Hamiltonian, or

$$U'(c) - q = 0 \quad (74)$$

so that

$$q = U'(c). \quad (75)$$

Also, since $q = W_k$ we have the condition that the marginal utility of consuming must equal the marginal utility of investing.

- Then $\dot{q} = 0$ yields the equation $f'(k) = \rho$ which is independent of q while from the second co-state equation $\dot{k} = 0$ yields $f(k) - c = g(q)$ where g is the inverse of U' . These are analogous to the equations we studied in the discrete time model and lead to the same phase diagram.