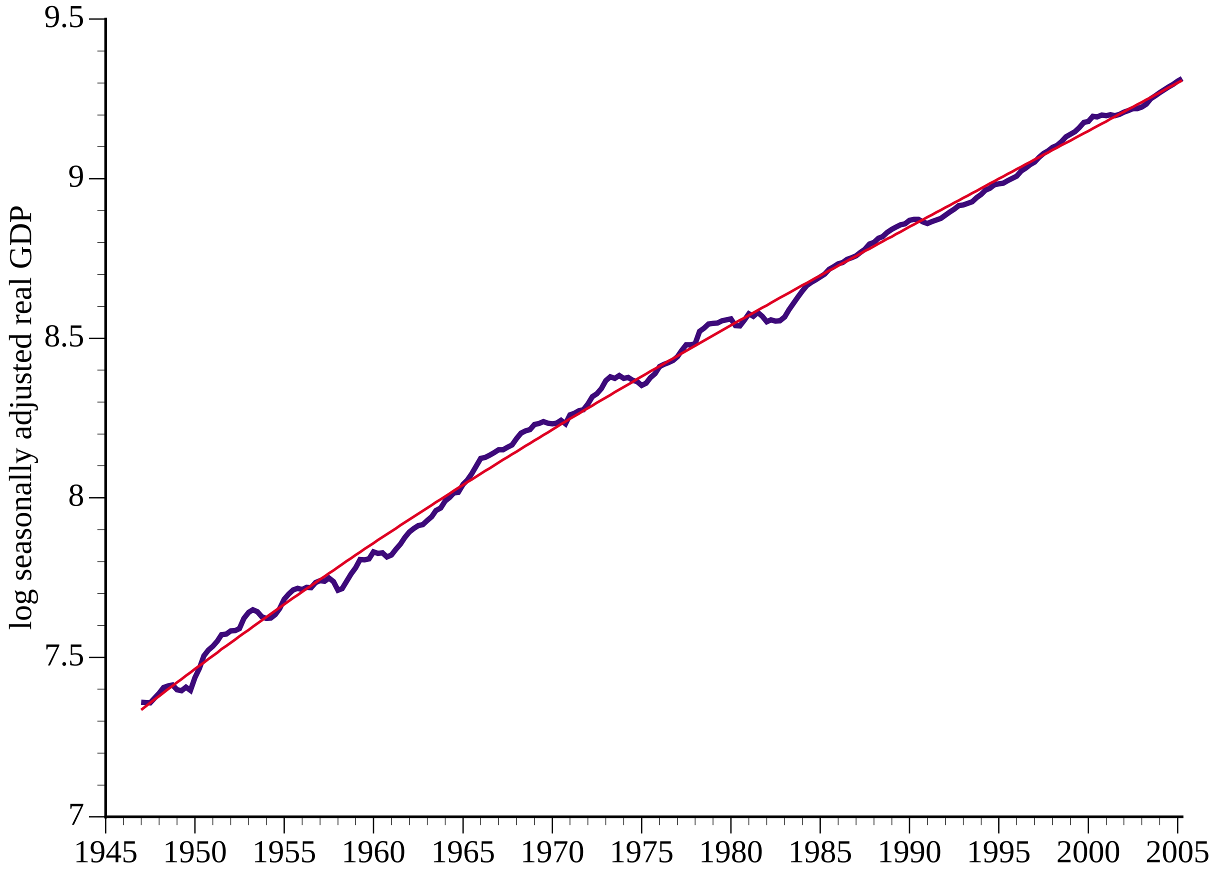
1. Why study Economic Growth?

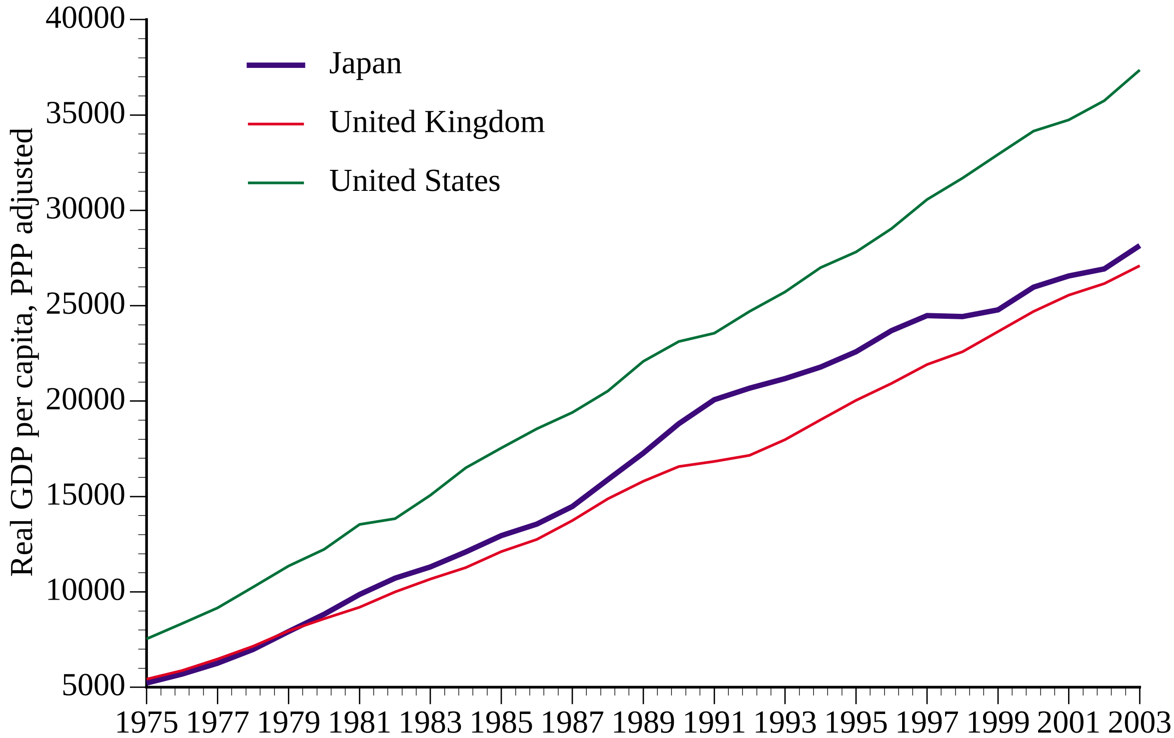
- The following graph shows the growth of US real GNP over the second half of the 20th century. Specifically, it is a plot of the log (seasonally adjusted real output) against time:



The graph also includes a cubic trend line. The linear term is 0.0434 (estimated standard error of 0.0011). This suggests and average annual real growth rate of around 4%. Both the quadratic (estimated value -0.00026) and cubic (estimated value 1.73E-06) terms are significantly different from zero at conventional significance levels (t-statistics of -5.89 and 3.45). Thus, there is evidence consistent with the hypothesis that economic growth trended down in the 1970s and 1980s relative to the 1960s and 1990s. An alternative way of allowing for lower trend growth rates in the 1970's and 1980's is to fit a piece-wise linear time trend with a break in the slope in the early 1970's and another in the early 1990s.

- The next few sets of notes focus on models of the longer run growth process in a market economy. We are interested in these models for several reasons:

  1. There can be quite large differences between the average economic growth rates of different countries for a considerable period of time (e.g. compare the post-WWII experiences of the UK, US and Japan) and for the same country over time. We would like to be able to explain why trend rates of growth vary over time.



  2. Such differences in economic growth rates seem to have important implications for economic welfare.

  3. The effects of growth rates is arguably more important than the effects of business cycles.

  4. Most market economies showed a downward trend in growth rates in the 1970s and 1980s and we would like to understand the reasons for this. What, if anything, can be done to obtain the relatively good growth rates of most of the 1960s and 1990s?

  5. Growth models form the basis of an important class of models of the business cycle, and the understanding of business cycles is a major focus of the course

  6. The *techniques* used in analyzing growth models also have application in other models of

business cycles but arise in a simpler context when considering growth. In particular, these models allow us to discuss representative agent and the use of the fundamental welfare theorems in a dynamic context.

2. The Aim of the Analysis

- We would like a theory of growth to explain why the economy has grown in the industrial era. (For most of human existence it has not been typical to encounter growth in access to material goods, or services supplied by others). More specifically, we would like our theory to explain why this rate has been around 3-4% p.a. over this period. At the same time we would like the theory to account for the broad outlines of other features of aggregate economic performance such as the change in real wages, the growth in employment and variations in the rate of inflation.

- An issue we discuss again later is whether essentially the same model we use to describe growth can also account for the evident variations in growth decade by decade or whether an alternative model is required to account for the short run fluctuations in growth rates year by year or quarter by quarter.

3. Factor Supply and Growth

- The most basic economic model of growth focuses on the growth in population, resulting in an increase in the labor force on the one hand, and saving leading to an increase in the capital stock on the other hand. Later we shall also discuss the role of resources, or physical and biological endowments, in the growth process. One way of interpreting “capital” and “labor” in this analysis is to see “capital” as any factor of production that can be augmented through market activity and “labor” as any factor of production with an exogenous rate of growth that cannot be changed by using market outputs.

- For explaining growth in per capita output we focus on capital accumulation as a means of increasing worker productivity (or relieving resource scarcity). In turn, security of property rights (and a stable and effective legal and law enforcement system) appears to be an important factor in producing capital accumulation.

- In *The Wealth of Nations* Adam Smith identified changes in the organization of production as being central to economic growth. In particular, he argued that the incentive for any one individual to specialize depended on the degree of specialization present elsewhere in the economy, and that specialization aided productivity, presumably as a result of "learning by doing." A problem with specialization is that it makes one more vulnerable to disruptions of the economy. Hence, political and social instability might be a factor limiting growth in many countries. We will not discuss the issue of specialization because it introduces additional mathematical complexities into the model. In practice, however, it is likely to be a significant factor behind economic growth.

- More recently, economists have also focused on human capital accumulation (either through education or learning by doing) and technological progress as an important factors in increasing worker productivity.

- We shall look at some simple models of the growth process which focus on the capital accumulation (or saving) decision. We ignore some of the above issues relating to property rights and specialization. A good model of economic development would need to take these issues into account.

4. Saving and Investment

- Suppose output at time t can be either consumed or invested (saved). Investment will have the benefit of increasing the output available for consumption in the future, but the cost of reducing current consumption.

- Assume labor supply does not depend on the real wage, so that labor supply grows along with the total population independently, or exogenously, with respect to output. Later we shall examine a model with variable labor supply.

- In the simplest model we also assume that a fixed fraction of income is saved, although later we shall examine the consequences of determining the level of saving through maximizing behavior. We shall also use the maximizing model as a basis for thinking about equilibrium *market* behavior.

- We assume aggregate output can be represented by the output of a representative firm:

$$Y_t = F(K_t,N_t)$$

which has constant returns to scale. Here K_t is the current aggregate capital stock and N_t the current aggregate labor supply. We assume F = 0 if either K = 0 or N = 0; we also assume F_K, $F_N > 0$, $F_{KK} < 0$, $F_{NN} = < 0$ and $F_{KN} > 0$.

- Because of the inelastic labor supply assumption, the labor force is proportional to the population. We therefore assume the labor supply is growing exogenously at the population growth rate $\lambda > 0$:

$$N_{t+1} = (1 + \lambda)\, N_t$$

- We assume capital depreciates at the rate $\delta > 0$ and denote by I_t the level of gross investment in period t. The evolution of the capital stock is then described by the equation:

$$K_{t+1}= (1-\delta)\, K_t + I_t$$

- In turn, gross investment is determined by the level of savings:

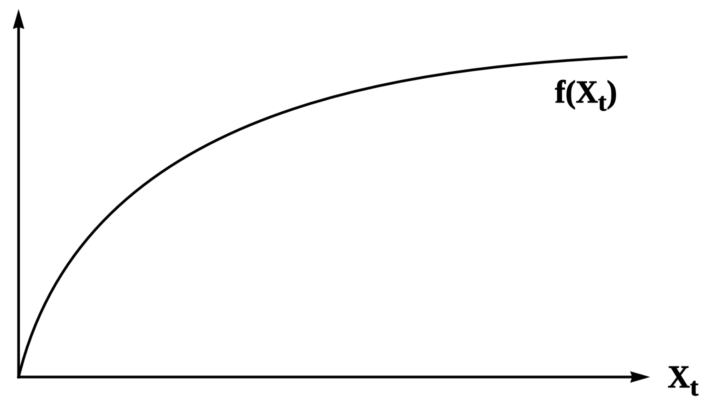
$$I_t = sY_t$$

- This completes the description of the basic assumptions of the model. Now let us see what they imply.

- First, use the fact that F is homogeneous of degree 1 to conclude:

$$Y_t / N_t = F(K_t/N_t, 1) \equiv f(X_t)$$

Note that $f' = F_K > 0$, $f'' = F_{KK} < 0$ and $f(0) = 0$ or in terms of a picture:



- Using the equations governing capital accumulation and labor force growth we get:

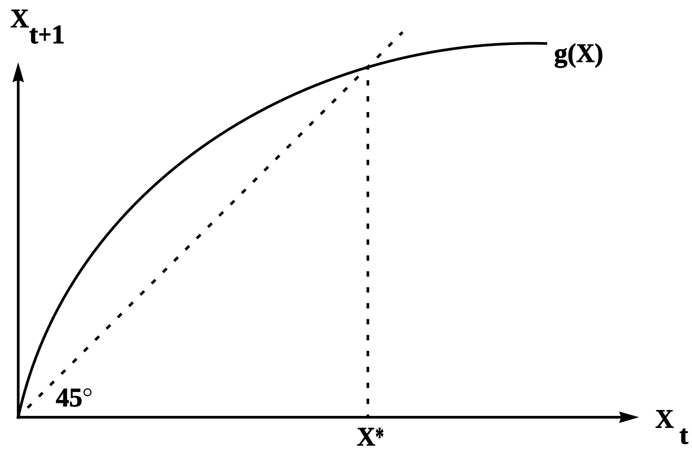
$$X_{t+1} = K_{t+1}/N_{t+1} = [sY_t + (1 - \delta)K_t]/[(1 + \lambda)N_t]$$

$$= [s/(1 + \lambda)]\ [Y_t / N_t] + [(1 - \delta)/(1 + \lambda)]\ X_t$$

which can be written in terms of f(.) as:

$$X_{t+1} = [s/(1 + \lambda)]\ f(X_t) + [(1 - \delta)/(1 + \lambda)]\ X_t \equiv g(X_t) \quad (1)$$

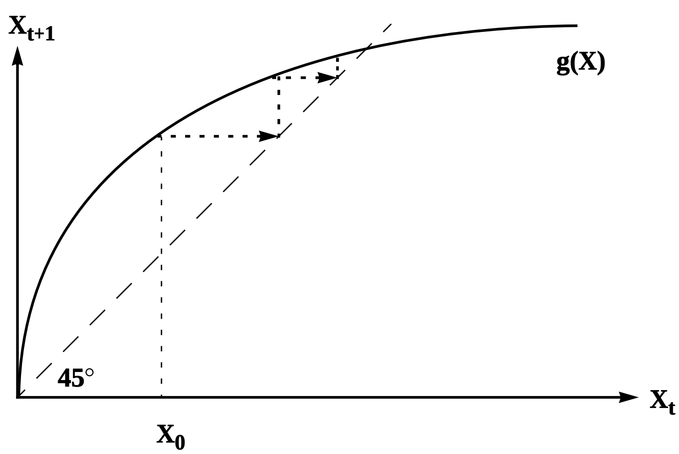
This relationship is presented in the graph below. Since $0 < s < 1$ and $1+\lambda > 1$, the first expression on the RHS of (1), $[s/(1+\lambda)]f(X_t)$ is a fraction of f(X). The second expression $[(1-\delta)/(1+\lambda)]X_t$ is a straight lie through the origin with a positive slope $[(1-\delta)/(1+\lambda)]$. The function g(X) is the sum of these two expressions.



5. Difference Equations

- Equation (1) is a first order non-linear homogeneous difference equation in X_t. We solve it by looking at the function g(.), which is the sum of a positive multiple of f(.) and a linear function of X_t as graphed above.

- Suppose X_t starts out at X_0



The value of X in period 1 will exceed X_0 because g(.) lies above the 45° line at X_0. As illustrated, X will approach X* monotonically with smaller and smaller steps each period. Similarly, if X_0 starts out above X*, X will decline back to X* in the long run.

- At X*, X* = [s/(1+λ)] f(X*) + [(1-δ)/(1+λ)] X*, or, rearranging terms,

$$(\lambda+\delta)\, X^* = s\, f(X^*) \tag{2}$$

The LHS of (2) represents the replacement of K necessary to make provision for depreciation and population growth. The RHS represents savings per capita.

- The above diagram implicitly assumes g'(0) > 1. Observe that

$$\lim_{x \to 0} g'(x) = \frac{1-\delta}{1+\lambda} + \frac{s}{1+\lambda} f'(0) > 1 \text{ if } f'(0) > \frac{\lambda+\delta}{s}$$

so the productivity of capital when the current stock of it is zero has to be sufficiently large if growth is to “take off.”

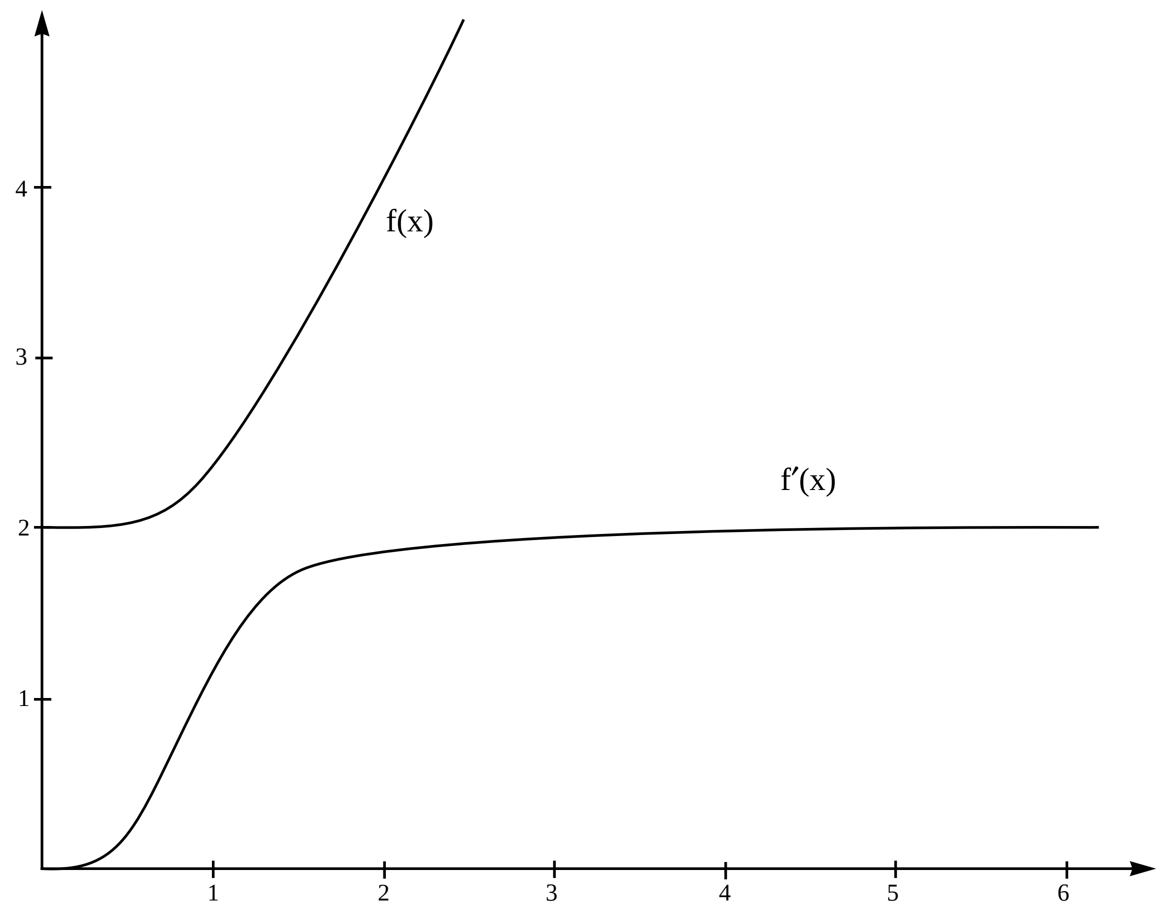
- The above diagram also is drawn on the assumption that g eventually crosses the 45° line. This will follow if $\lim_{x \to \infty} g'(x) < 1$; that is, $\lim_{x \to \infty} f'(x) < \frac{\lambda+\delta}{s}$. For example, if f is bounded we will have $\lim_{x \to \infty} f'(x) = 0$. A common constant returns to scale production function where this condition need not hold is the constant elasticity of substitution function:

$$F(K,N) = A(K^{1/e} + BN^{1/e})^e;\ \text{with } e > 0$$

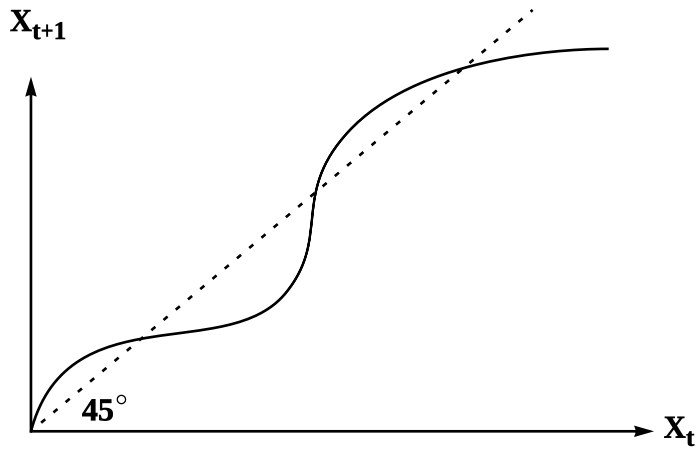
where $f(x) = A(B + x^{1/e})^e$. In particular, note that

$$f'(x) = A(B + x^{1/e})^{e-1}x^{1/e-1} = A(B/x + 1)^{e-1} \to A \text{ as } x \to \infty.$$

For e = 1/4, A = 2 and B = 1 this function is graphed below.



- The values 0 and X* for X solve g(X) = X and are known as fixed points of g(.). If the initial value of X, X_0 = X* then X_t = X* for all t. Similarly, if X_0 = 0 then X_t = 0 for all t. The values 0 and X* for X_t are therefore also known as stationary points of the difference equation. More complicated f(.) functions will produce more stationary points as can be seen from the following diagram:



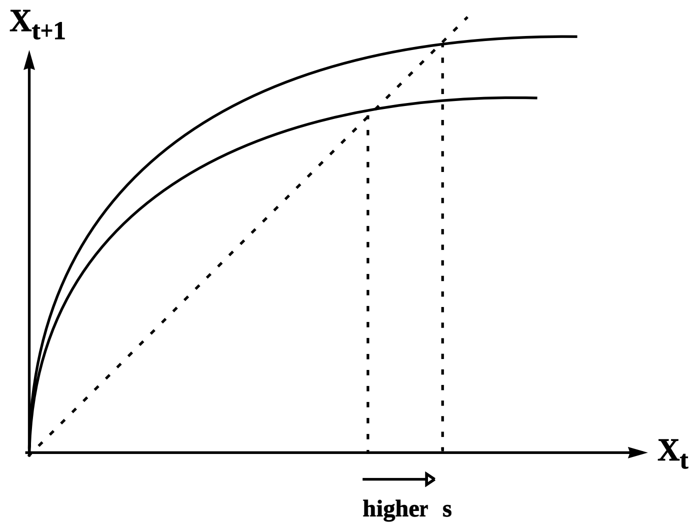
Some f(.) functions, such as the one produced by a CES production function with a large value of A, may produce no stationary points – a point to which we shall return below.

6. "Comparative Statics" in a Dynamic Model

- We can do some comparative dynamics by asking what happens to X* and the path of X as the parameters in the model change:

  - given X = X* in the long run equilibrium, the capital stock and hence output will grow at the rate λ. Thus, part of the reason for the relatively high growth rates of Japan in the 1960's was that the opportunity to move large number of workers out of agriculture into manufacturing (through the use of more modern farming techniques) greatly increased the labor force growth rate in manufacturing and allowed output to expand rapidly in that sector of the economy. A similar phenomenon is happening in China and India today.

  - changes in the saving rate s do not alter the long run growth rate of output. Increases in the saving rate s will, however, make X* larger and X_t larger at each t - output per head will therefore also be larger in the long run equilibrium of an economy with a higher savings rate.



Algebraically, we can use calculus to find

$$(\lambda + \delta)\frac{dX^*}{ds} = f(X^*) + sf'(X^*)\frac{dX^*}{ds}$$

that is,

$$\frac{dX^*}{ds} = \frac{f(X^*)}{\lambda + \delta - sf'(X^*)} = \frac{X^*f(X^*)}{s[f(X^*) - X^*f'(X^*)]} \tag{3}$$

But since F is homogeneous of degree 1:

$$F(\mu K, \mu N) = \mu F(K, N) \text{ for } \mu > 0 \tag{4}$$

Differentiate (4) with respect to μ to get:

$$KF_K + NF_N = F(K, N); \tag{5}$$

(The result (5) is known Euler's theorem.) Then we have

$$f'(X) = F_K = (F - NF_N)/K = (f - F_N)/X$$

or, rearranging,

$$f(X_t) - X_t f'(X_t) = F_N > 0. \tag{6}$$

Hence, from (3) and (6) we conclude that an increase in the savings rate will raise the long stationary per capita capital stock.

- what happens to consumption as s changes? If s = 0, we end up at X = 0 so that consumption = 0 in the long run equilibrium. If s = 1, we also have zero consumption.

7. *The Golden Rule*

- Presumably, some s, 0 < s < 1, is "best". Suppose we choose s (and therefore X*) to maximize consumption at X*. This is equivalent to choosing s to

$$\max_{X^*}(1-s)f(X^*) \text{ subject to } sf(X^*) = (\lambda+\delta)X^*$$

Note that given the constraint, choosing s is equivalent to choosing X*. The constraint implies $(1-s)f(X^*) = f(X^*) - (\lambda+\delta)X^*$ so the maximizing X* will be given by

$$f'(X^*) = \lambda + \delta \tag{7}$$

This leads to the so-called "golden rule" savings rate

$$s = X^* f'(X^*)/f(X^*) = \text{elasticity of } f(.) \tag{8}$$

8. *Factor Payments*

- What happens to factor payments along the equilibrium path? The rental on capital is given by $F_K = f'(X)$ and since we have assumed $f'' < 0$, rental rates decrease along the equilibrium path until X* is reached. The real wage will be given by F_N. Using the assumption that F is homogeneous of degree one, we showed above that

$$F_N = \text{real wage} = f(X_t) - X_t f'(X_t) \tag{9}$$

Then the derivative of the real wage with respect to X_t is

$$f'(X_t) - f'(X_t) - X_t f''(X_t) = - X_t f''(X_t) \quad (10)$$

and since $f'' < 0$ the real wage is increasing along the equilibrium path.

9. Performance of the Model

- How well does this model do empirically?

- At a trivial level, we can associate "developed" economies with economies having a "more productive" technology or a higher level of output F from a given level of inputs. These economies also have a higher level of capital per capita (X) - particularly if we include human capital in K.

- Perhaps, as economies "mature", their economic growth rate slows down as predicted by the model. There is a large empirical literature on the so-called "convergence hypothesis" which postulates that per capita GNP growth rates are negatively related to the initial level of per capita GNP so that economies tend to "converge" over time in terms of per capita GNP levels. It has been claimed that this hypothesis is consistent with data from the states of the US, regions of Western Europe, as well as across many samples of "similar" countries ("convergence clubs").

- On a recent research project, we estimated an equation for per capita GDP growth using World Bank data from 173 countries (the graph of Japan, the UK and the US above came from that data). Although the maximum sample size for any one country was 52 years, there was an average of 37.6 years of data for each country. The estimated equation was

$$\hat{y}_{it} = c_i + a_1 \hat{y}_{it-1} + a_2 (1/\ln y_{it-1}) - a_3 (\hat{y}_{it-1}/\ln y_{it-1}) - a_4 (\ln y_{it-1} - \ln y_{USt-1})$$

where the terms c_i are country specific constants. The parameter estimates nd their corresponding estimated standard errors were

Parameter	Estimate	Std. error
a_1	0.9362	0.0886
a_2	0.9431	0.1178
a_3	–6.1930	0.7009
a_4	–0.0152	0.0030

The negative coefficient on the difference between country i GDP per capita and US GDP per capita implies that per capita growth rates of other countries will tend to converge toward those of the United States over time. The positive coefficient on the inverse log level of per capita GDP ($1/\ln y_{it}$) implies that growth rates will tend to diminish as per capita GDP increases. Furthermore, the negative coefficient on the interaction term implies that growth rates will tend to become more persistent as the economy matures.

- The estimated model sits uncomfortably with the conclusion that growth in per capita terms will cease in the long run. The estimates imply growth will *slow down* over time, but it will be a very long time before growth ceases. It may not even cease, since the level of GDP where that happens is way beyond the values included in the estimation sample.

- It might also be argued that some high growth countries have a higher marginal propensity to save, s, leading to higher growth rates at the same level of X and eventually a higher level of X*, even though the model predicts that all economies, regardless of their level of per capita savings s, would end up with *no* long run growth in *per capita* terms. The effect of different savings rates in a model with international capital flow is, however, a more complicated issue and is left to a homework problem.

- At a more quantitative level, if we can describe aggregate production with the function $Y = F(K,N)$, then we will have (differentiating with respect to time and dividing through by Y):

$$\hat{Y} = \frac{KF_K}{Y}\hat{K} + \frac{NF_N}{Y}\hat{N} = s_K\hat{K} + s_N\hat{N}$$

where ^'s denote percentage rates of change and the s_i are factor shares in aggregate income. The growth rate of output is the weighted growth rate of the inputs with weights given by the relevant factor shares. Under constant returns to scale, the weights sum to 1 and can be obtained from the National Accounts. For the US over a long period we find the annual growth rate of output is about 3.5%, labor's share about 0.75, capital's share about 0.25, the net growth of the capital stock about 2.5% and the% change in total hours worked about 1.25%. The weighted sum is about $^1/_2$ the recorded average annual growth rate.

- We can call the difference between the two numbers "exogenous technological change" but this is not useful unless we have a theory which leads to some alternative way of understanding technological change. Recent papers have attempted to model technological change by looking at R&D races and models of emulation of technological developments by other firms. Since technological innovation can be emulated, firms have a reduced incentive to invest in R&D. Patents are one way of coping with this externality, but at the cost of allowing monopoly production for a period of time. Another problem with patents is that they encourage "wasteful" R&D which merely attempts to duplicate existing patents but which has no, or little, social value.[1]

- Other recent models of the growth process have asserted that government-provided public goods are an omitted variable from the aggregate production function and may account for some of the "missing" factors of production. In particular, some authors have claimed that at least part of the slow-down in growth in the US in the 1970's and early 1980's can be explained by a fall in government investment in infrastructure. This claim has been supported not only with time series

1. An alternative method of increasing R&D is to subsidize it directly. However, it is questionable whether government bureaucrats have sufficient information to choose the best technologies to develop – or whether politicians would let them choose the best as determined by some "objective" criterion rather than, for example, those investment projects located in the most marginal electorates. Using patents and privately funded R&D, the firms at least have an incentive to invest in research which is likely to benefit consumers.

evidence on aggregate government spending on capital projects but also by splitting such expenditure into components for administrative buildings, courts, hospitals, defence equipment and transport, public utility and communications infrastructure. Only the transport, public utility and communications components have been shown to be significantly positively correlated with economic growth rates. Furthermore, changes in government "consumption" expenditures, such as transfer payments, have been shown to have zero or negative correlations with private sector growth. Other recent papers have also related cross-sectional variations in growth rates in the US to differences in infrastructure spending by *State and local* governments. There are also some papers claiming to have found contrary results. A particularly difficult issue to answer empirically is the cause and effect relationship between government infrastructure spending and economic growth – does additional spending in rapidly growing regions cause, or result from, the high rate of private sector growth? Attempts to extend these US results to other economies have found less significant positive results than were found for the US. However, broad international evidence (including less developed economies in the sample) would seem to be consistent with the notion that inadequate provision of public infrastructure, including an independent legal system guaranteeing private property rights, is one reason for under-development in many nations.

- We might be able to adjust for the quality of N by measuring schooling and other investment in human capital. We might also attempt to adjust the physical capital stock for quality by measuring R&D input into its production. Some authors have argued that the return to investment in education depends on the level of education of other individuals in the society. As the level of education rises individuals find it more profitable to spend more time in human capital investment. This introduces changes that fundamentally alter the properties of the model. In fact, these models are one type of so-called "endogenous growth" model.

10. Endogenous growth

- The key aim of the endogenous growth models is to explain why *per capita* growth might continue *indefinitely*. The simple growth model discussed above has the capital stock and output ultimately growing at the same rate as population - so there is no growth in per capita terms in the long run stationary equilibrium.

- From a mathematical perspective, the key requirement to ensure continual growth in per capita terms is that the marginal product of capital does not decline to $(\lambda+\delta)/s$ as capital is accumulated. For example, a production function like the CES one discussed above might leave capital sufficiently productive as the capital labor ratio rises that $f'(X)$ doesn't decline to equal $(\lambda+\delta)/s$ no matter how large X is. There will be no fixed point of the first order difference equation and per capita capital X, and therefore per capita output and per capita consumption, will continue to grow without bound.

- Ljungqvist and Sargent (chapter 11) discuss a number of economic mechanisms for obtaining perpetual growth. We briefly discuss these within our framework.

Externality from spillovers

- In this model, the R&D done by some firms is assumed to positively affect the productivity of other firms, but in a way that cannot be captured. We distinguish the capital under the control of the representative firm K_t from the "economy-wide" average physical capital per worker $\bar{k}_t$. Specifically, we assume that aggregate output is now given by:

$$Y_t = F(K_t, \bar{k}_t N_t)$$

for F constant returns to scale with an additional equilibrium condition

$$\bar{k}_t = K_t/N_t$$

- If we proceed as above, we find

$$Y_t/N_t = F(K_t/N_t, \bar{k}_t) = F(X_t, \bar{k}_t)$$

which *in equilibrium* can also be written

$$Y_t/N_t = X_t\, F(1,1) = X_t f(1)$$

- The difference equation for X_t now becomes

$$X_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{sY_t + (1-\delta)K_t}{(1+\lambda)N_t} = \frac{s}{1+\lambda}F(X_t, \bar{k}_t) + \frac{1-\delta}{1+\lambda}X_t \qquad (11)$$

and once again in equilibrium equation (11) becomes

$$X_{t+1} = \frac{s}{1+\lambda}f(1)X_t + \frac{1-\delta}{1+\lambda}X_t = \left[\frac{s}{1+\lambda}f(1) + \frac{1-\delta}{1+\lambda}\right]X_t \qquad (12)$$

- The difference equation (12) passes through the origin. If it has a slope > 1, the economy will grow forever, while if the slope < 1, the economy will decay back to zero capital. The necessary and sufficient condition for perpetual growth is then

$$sf(1) + 1 - \delta > 1 + \lambda, \text{ that is } \qquad f(1) > \frac{\lambda + \delta}{s}$$

- If this economy does grow, its growth rate $X_{t+1}/X_t - 1$ is independent of t:

$$\frac{X_{t+1}}{X_t} - 1 = \frac{sf(1) - (\lambda + \delta)}{1 + \lambda}$$

All factors reproducible

- Allowing all factors of production to be “produced” (e.g. by allowing investment to augment the amount of human capital per worker) is yet another means of achieving continuous per capita growth. While this might seem plausible, we should note that with finite-lived individuals the human capital has to be of a form that can be passed on to subsequent generations. “Scientific knowledge” might have this cumulative effect on labor productivity.

- Consider the following simple example. Suppose aggregate total output can be described by an aggregate production function of the form

$$Y_t = K_t^{\alpha} N_t^{1-\alpha}; 0 < \alpha < 1 \tag{13}$$

where Y_t = aggregate output, K_t = aggregate capital services and N_t = aggregate labor *services*. Assume that without investment in physical capital, capital services would depreciate at the rate δ per period and that, without investment in human capital, labor services would grow at the rate λ per period (due to population growth alone with a fixed ratio of employees to population and fixed labor hours per employee).

- As in the simple Solow-Swan growth model, assume households invest a constant proportion s_1 of their income in physical capital accumulation, but now assume also that they invest a constant proportion s_2 of their income in augmenting their labor productivity through education or training.

- Specifically, assume capital and labor services grow according to the difference equations

$$K_{t+1} = (1-\delta)K_t + I_t; 0 < \delta < 1 \tag{14}$$

$$N_{t+1} = (1+\lambda)N_t + H_t; 0 < \lambda < 1 \tag{15}$$

with

$$I_t = s_1 Y_t; 0 < s_1 < 1 \tag{16}$$

and

$$H_t = s_2 Y_t; 0 < s_2 < 1 \tag{17}$$

Define $k_t = K_t/N_t$. From the difference equations (14) and (15) for K_{t+1} and N_{t+1} we obtain

$$X_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{(1-\delta)K_t + I_t}{(1+\lambda)N_t + H_t}$$

Then from (16) and (17), the production function (13), and after dividing through by N_t we get

$$X_{t+1} = \frac{(1-\delta)X_t + s_1 X_t^{\alpha}}{(1+\lambda) + s_2 X_t^{\alpha}} = g(X_t) \tag{18}$$

- Now we can show $dX_{t+1}/dX_t > 0$ for all $X_t > 0$:

$$\frac{dX_{t+1}}{dX_t} = \frac{(1+\lambda+s_2X_t^{\alpha})(1-\delta+\alpha s_1 X_t^{\alpha-1}) - [(1-\delta)X_t + s_1X_t^{\alpha}]\alpha s_2 X_t^{\alpha-1}}{(1+\lambda+s_2X_t^{\alpha})^2} \tag{19}$$

$$= \frac{(1+\lambda)(1-\delta) + (1-\alpha)(1-\delta)s_2X_t^{\alpha} + (1+\lambda)\alpha s_1 X_t^{\alpha-1}}{(1+\lambda+s_2X_t^{\alpha})^2}$$

Also $X_t = 0$ is a stationary point of the difference equation since, if we substitute $X_t = 0$ on the right hand side of (18), we obtain $X_{t+1} = 0$. Also, if we substitute $X_t = 0$ on the right hand side of (19) we obtain as $X_t \to 0$, (and using $\alpha < 1$):

$$\frac{dX_{t+1}}{dX_t} = \frac{(1+\lambda)(1-\delta)X_t^{1-\alpha} + (1-\alpha)(1-\delta)s_2X_t + (1+\lambda)\alpha s_1}{X_t^{1-\alpha}(1+\lambda+s_2X_t^{\alpha})^2} \to \frac{(1+\lambda)\alpha s_1}{X_t^{1-\alpha}(1+\lambda)^2} \to \infty$$

We conclude that $X_t = 0$ is an unstable stationary point of the difference equation and the physical capital/human capital ratio will diverge away from $X_t = 0$.

- Equation (19) also implies

$$\frac{dX_{t+1}}{dX_t} = \frac{(1+\lambda)(1-\delta)X_t^{-2\alpha} + (1-\alpha)(1-\delta)s_2X_t^{-\alpha} + (1+\lambda)\alpha s_1 X_t^{-\alpha-1}}{[(1+\lambda)X_t^{-\alpha} + s_2]^2} \to 0 \text{ as } X_t \to \infty$$

and we conclude that the difference equation has at least one stationary point.

- When X_t is stationary at X*, the difference equation (18) becomes

$$X^* = \frac{(1-\delta)X^* + s_1 X^{*\alpha}}{(1+\lambda) + s_2 X^{*\alpha}} \tag{20}$$

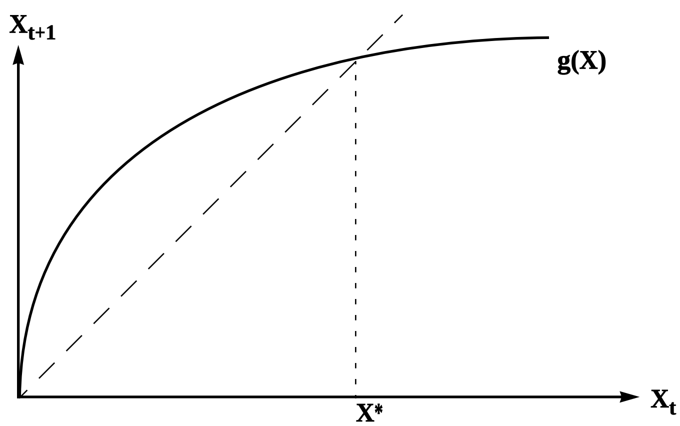
Equation (20) can be re-arranged to get

$$(1+\lambda)X^* + s_2 X^{*\alpha+1} = (1-\delta)X^* + s_1 X^{*\alpha}$$

or

$$(\lambda + \delta)X^{*1-\alpha} = s_1 - s_2 X^* \tag{21}$$

Since $\alpha < 1$, the left side of (21) is monotonic increasing in X*, while the right side is monotonic decreasing in X*. In addition, the left side equals 0 at X* = 0, while the right side is $s_1 > 0$ at X* = 0. Thus, (21) has a unique positive solution for the stationary physical capital/human capital ratio X*.



- We have shown that the phase diagram for the first order non-linear difference equation for X_t is as illustrated in the diagram. The function g(X) starts out above the 45° line and has exactly one fixed point at X*. Hence, X_t will converge to X* from any $X_0 > 0$.

- At the stationary point X*, the ratio of K to N is fixed so K and N must therefore grow at the same rate. The economy is said to be on a *balanced growth path*. With K and N growing at the same rate, output Y also grows at that rate. Let the growth rate of N (and K and Y) in the stationary equilibrium be γ. From the difference equation (15) we obtain:

$$\gamma_t = \frac{N_{t+1} - N_t}{N_t} = \lambda + \frac{H_t}{N_t} \qquad (22)$$

But from the production function (13) and the investment equation (17)

$$\frac{H_t}{N_t} = \frac{s_2 K_t^{\alpha} N_t^{1-\alpha}}{N_t} = s_2 X_t^{\alpha} \qquad (23)$$

In the stationary equilibrium, $X_t = X^* > 0$. Substituting into (23) and then into (22) we find that the stationary growth rate of N, and hence K and Y, exceeds the population growth rate λ:

$$\gamma = \lambda + s_2 X^{*\alpha} \qquad (24)$$

This economy continues to grow in *per capita* terms *forever*. The ability to augment labor supplies through investment in human capital keeps the marginal product of capital from falling as more capital is accumulated so per capita growth does not cease.

- Using the fact that X* is an implicit function of the parameters s_1, s_2 and λ (given in (21)), we can differentiate expression (24) for the stationary growth rate to find that γ responds positively to an increase s_1, s_2 or λ. Increases in either saving rate, or the population growth rate λ therefore will increase the long run *per capita growth rate* of this economy.

- As in the Solow-Swan model, there is a unique positive stationary state in this economy that is globally stable. Unlike the Solow-Swan model, the economy continues to grow in *per capita* terms in the stationary state. Also unlike the simple model, increases in savings rates can have a *permanent* effect on the growth rate of the economy in this model. Finally, an increase in the population growth rate actually increases the per capita growth rate in the long run stationary state too. This comes about because the new generations are born endowed with the human capital we have already invested in. The "human capital" we are thinking of here is therefore like "scientific knowledge" or some other "disembodied" or "transferable" form rather than being embodied in the non-transferable skills of a current generation.

- Another attempt to explain endogenous growth focuses on increasing returns from specialization in production. Ljungqvist and Sargent discuss such a model on pages 292–297. but in an optimizing framework rather than the simple "mechanistic" model we are discussing here.

11. Limits to Growth?

- Endogenous growth models, or models with continuous technological change, attempt to explain the empirical phenomenon of continual growth in per capita output over extremely long periods of time. Many ecologists and other critics of market economies have argued there are limits to growth implied by the limited availability of natural resources and a limited ability of natural ecosystems to provide essential inputs and accommodate wastes produced by market processes.

- At a technical level, we allow energy, minerals and other natural resources to constitute another category of factors of production. We might then expect that limitations on the growth in supply of such natural resources would limit economic growth rates. That is, even if "effective" labor can be made to grow at rates above the population growth rate λ, so that the marginal product of capital does not decline as K/N grows, we might still have the marginal product of capital declining as natural resources become *relatively* more scarce over time.

- Of course, as the relative price of raw materials rises over time, firms would also have incentives to exploit resource deposits that are more marginal (e.g. re-work old mines or mine the sea bed), replace particular resources with substitutes (use aluminum instead of copper in electric wires, sand - fibre optics - instead of copper in phone lines etc.), increase the intensity of use of limited resources (e.g. by switching to hydroponic agriculture to save on scarce land, invest in recycling technologies, membrane and desalination technology to clean water, producing usable energy directly from sunlight and so forth).[2] Even with regard to energy, technologies currently available, such as solar cells, could replace fossil fuels if the latter had a sufficiently high price. The supply of solar energy is virtually inexhaustible. Nuclear and geothermal resources are also very large. Apart from using *physical* resources more efficiently, the productivity of natural *biological* resources can be augmented through genetic engineering and breeding programs.[3]

- As Ljungqvist and Sargent remark, the critical technical requirement for perpetual growth is that there must be a “core” of capital goods that is produced with constant returns technologies and without the direct or indirect use of non-reproducible factors. They discuss a simple case where labor is taken as the fixed exogenous resource and the sole capital good is produced without any input of the economy’s constant labor endowment.

- Specifically, assume that goods output is produced according to

$$Y_t = F(\phi K_t, N_t)$$

where $0 \le \phi \le 1$ is the fraction of capital employed in producing consumption goods, while capital goods now are produced entirely with capital. Keeping within the spirit of the simple Solow-

2. Furthermore, in practice services are becoming relatively more important over time as economies grow. A restaurant meal and a home cooked meal might use the same “raw material” inputs, but the restaurant meal provides greater value because of all the added services – the higher quality cooking, the atmosphere, the new taste experiences and so on. In other words, as economies grow, there is a tendency for the “resource intensity” of output to fall. In fact, the continual decline in the relative price of raw materials and agricultural goods suggests that “raw materials” are becoming *less* scarce over time.

3. This is not to say that there is no “environmental problem” associated with economic activity. There usually is, but the problem is not one of limited resources per se. Rather, the problem is one of incomplete property rights. Some resources are unowned, or communally owned, and so are over-exploited in a “tragedy of the commons.” Since the resources are unowned, they are used without charge and the environmental costs of producing market goods and services are not reflected in their prices. The solution is to introduce property rights so use of these resources is reflected in market prices and optimal trade-offs are made between use and preservation. Environmental degradation is a sign of inefficiency resulting from inadequate market structures, but is not necessarily evidence of environmentally-based “limits to growth.”

Swan model, we take ϕ as fixed. Thus, investment is given by

$$I_t = A(1-\phi)K_t$$

and once again capital accumulation follows

$$K_{t+1} = (1-\delta)K_t + I_t$$

- Now the difference equation for the capital/labor ratio becomes

$$X_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{[1-\delta+A(1-\phi)]K_t}{(1+\lambda)N_t} = \frac{1-\delta+A(1-\phi)}{1+\lambda}X_t$$

Thus, the per capita capital stock will grow so long as $A(1-\phi) > \delta+\lambda$, in which case the growth rate will be

$$\frac{A(1-\phi)-(\delta+\lambda)}{1+\lambda}$$

- Goods output per capita will be given by:

$$\frac{Y_t}{N_t} = F(\phi X_t, 1) = f(\phi X_t)$$

which will have a growth rate equal to the elasticity of f with respect to X times the growth rate of X.