Rice University

Answers to Mid-Semester Examination Fall 2004

ECON 501: Advanced Microeconomic Theory

Part A

1. Consider the following binary relation. Take $X = \mathbb{R}^L$, and define $x \succsim y$ if $x \geq y$.

   Show whether or not this relation is complete, transitive, strongly monotone, strictly convex.

   Answer: For this answer I take $L \geq 2$.

   (a) **It is not complete. Recall completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both). So consider x and y with $x_1 > y_1$ and $x_2 < y_2$. Then we have neither $y \succsim x$ nor $x \succsim y$.**

   (b) **It is transitive. Recall transitivity requires if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. Now $x \succsim y$ and $y \succsim z$ means $x \geq y$ and $y \geq z$. But $x \geq y$ and $y \geq z \Rightarrow x_\ell \geq y_\ell$ and $y_\ell \geq z_\ell$ for all ℓ. By transitivity of $\geq$ we thus have $x_\ell \geq z_\ell$ for all ℓ, and so $x \geq z$, that is, $x \succsim z$, as required.**

   (c) **It is strongly monotone. Recall strongly monotone requires if $x \geq y$ and $x \neq y$ then $x \succ y$.**

   Now $x \geq y$, and $x \neq y \Rightarrow x_\ell > y_\ell$, for all ℓ and $x_k > y_k$, for some k. Thus $x \geq y$, and $x \neq y$ implies $x \succsim y$ and NOT$(y \succsim z)$, that is, $x \succ y$, as required.

   (d) **It is strictly convex. Recall strict convexity requires if $x \succsim z$ and $y \succsim z$ and $x \neq z$ then $\alpha x + (1-\alpha) y \succ z$ for all $\alpha \in (0,1)$.**

   Now if $x \succsim z$ and $y \succsim z$ then $x_\ell \geq z_\ell$ and $y_\ell \geq z_\ell$ for all ℓ. And if $x \neq z$ then for some $k \in \{1, \ldots, L\}$ we have $x_k > z_k$. Thus for any α in $(0,1)$, $\alpha x_\ell + (1-\alpha) y_\ell \geq z_\ell$ for all ℓ, and $\alpha x_k + (1-\alpha) y_k > z_k$. Hence we have $\alpha x + (1-\alpha) y \geq z$ and $\alpha x + (1-\alpha) y \neq z$, and so, $\alpha x + (1-\alpha) y \succsim z$ and NOT$(z \succsim \alpha x + (1-\alpha) y)$, i.e., $\alpha x + (1-\alpha) y \succ z$, as required.

2. Cathy's aim is to raise as much money as possible in the T units of time available to her. She is able to allocate her time between two activities x_1 and x_2. Suppose both activities are 'equally productive' in generating money in the sense that if she spends $x \in [0, T]$ units of time in activity $i = 1, 2$, she will raise $f(x)$ dollars. Assume $f(.)$ is a strictly concave and strictly increasing function with $f(0) = 0$.

   State whether you agree or disagree with the following statement and (briefly) explain your reasoning.

   "The solution to Cathy's problem is obvious, she should spend $T/2$ units of time in each activity since that equalizes the amount she raises per unit of time in each activity: that is, if $x_1 = x_2 = T/2$, then we have $f(x_1)/x_1 = f(x_2)/x_2$"

Disagree. Cathy's aim is to maximize the sum of monies raised from each activity subject to her time constraint. That is,

$$\max_{\langle x_1, x_2 \rangle} f(x_1) + f(x_2) \text{ s.t. } x_1 + x_2 \leq T.$$

Forming the Lagrangian

$$\mathcal{L} = f(x_1) + f(x_2) - \lambda (x_1 + x_2 - T)$$

First order conditions are

$$\begin{aligned} f'(x_1) &= \lambda \\ f'(x_2) &= \lambda \end{aligned}$$

So solution characterized by $f'(x_1) = f'(x_2)$ which implies $x_1 = x_2$. And since $x_1 + x_2 = T$, we get $x_1 = x_2 = T/2$. It is true that this equalizes the amount she raises per unit of time, but that is not why this is an optimal allocation. It is optimal because it equalizes the marginal amount of money raised from the last unit of time spent in each activity.

3. *State the Weak Axiom of Revealed Preference and draw a diagram illustrating its implication in a two-good world for the change in consumption of an individual due to a change in prices.*

   If $x, y \in B$ and $x \in C(B)$, and if $x, y \in B'$ and $y \in C(B')$ then $x \in C(B)$. In words, if x and y are both available in choice problems 1 and 2, and if x was an element of the chosen set in the first problem and y was an element of the chosen set in the second problem, then x must also be an element of the chosen set in the second problem. Or in terms of the UMP:

$$\begin{aligned} \textbf{if } p.x(p', w') &\leq w \textbf{ then } p'.x(p, w) \geq w \\ \textbf{and if } p.x(p', w') &< w \textbf{ then } p'.x(p, w) > w \end{aligned}$$

   [Some of you quoted the definition from the textbook which assumes that $x(p, w)$ is a function in which case WARP may be expressed as follows:

$$\textbf{If } p.x(p', w') \leq w \textbf{ and } x(p', w') \neq x(p, w) \textbf{ then } p'.x(p, w) > w.$$

   This is fine as well, but do be aware of why we have the simpler condition for the case in which $x(p, w)$ is a function and not a correspondence.]

4. *Explain the meaning of the identity*

$$x_l(p_1, p_2, e(p_1, p_2, u)) \equiv h_l(p_1, p_2, u)$$

   where x_l is the uncompensated demand for good l, h_l is the compensated demand for good l and e is the expenditure function.

   Using this identity derive the Slutsky equation and the 2×2 substitution matrix.

The uncompensated demand for good ℓ given prices and given the mininum expenditure needed at those prices to achieve the utility target u, is identically equal to the compensated demand for that good facing the same prices and for the utility target u.

$$\frac{\partial x_\ell}{\partial p_k} + \frac{\partial x_\ell}{\partial w} \times \underbrace{\frac{\partial e}{\partial p_k}}_{=h_l \text{ (by Shephard's lemma)}} = \frac{\partial h_\ell}{\partial p_k}$$

So Slutsky equation

$$\frac{\partial x_\ell}{\partial p_k} = \underbrace{\frac{\partial h_\ell}{\partial p_k}}_{\text{substitution effect}} - \underbrace{x_\ell \frac{\partial x_\ell}{\partial w}}_{\text{income effect}}, \text{ since } h_l(p_1,p_2,u) = x_l(p_1,p_2,e(p_1,p_2,u))$$

And

$$S = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial w} & \frac{\partial x_2}{\partial p_1} + x_2 \frac{\partial x_2}{\partial w} \\ \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial w} & \frac{\partial x_2}{\partial p_2} + x_2 \frac{\partial x_2}{\partial w} \end{bmatrix}$$

Part B

5. Consider a household that is seen to purchase quantities of just two goods, bread and cheese. Denote quantities of bread by x and quantities of cheese by y. The household comprises two individuals; Andrew, whose preference relation can be represented by the utility function $u_A(x,y) = x$ and Brenda, whose preference relation can be represented by the utility function $u_B(x,y) = y$.

   (a) *Derive the uncompensated demand functions for both Andrew and Brenda and their indirect utility functions.*

   $$\max_{\langle x,y\rangle} u_A(x,y) = x \text{ s.t. } p_x x + p_y y \leq w_A$$

   Solution is to spend all wealth on x and buy no y (since y doesn't figure in Andrew's utility. Similarly Brenda maximizes her utility by spending all her wealth on y (since x doesn't appear in her utility). Thus

   $$\begin{aligned} x_A(p_x,p_y,w_A) &= \frac{w_A}{p_x},\ y_A(p_x,p_y,w_A) = 0 \\ x_B(p_x,p_y,w_B) &= 0,\ y_B(p_x,p_y,w_B) = \frac{w_B}{p_x} \end{aligned}$$

   (b) The households' wealth w is divided evenly between Andrew and Brenda. **Suppose that you observe the aggregate demands of this household and you interpret it as if it came from just a single consumer.**
   Find the demands of the supposed single consumer.

If a positive representative consumer exists, its demand will equal aggregate demand, that is:

$$\begin{aligned} x(p_x,p_y,w) &= x_A(p_x,p_y,w/2) + x_B(p_x,p_y,w/2) = \frac{w}{2p_x} + 0 \\ y(p_x,p_y,w) &= y_A(p_x,p_y,w/2) + y_B(p_x,p_y,w/2) = 0 + \frac{w}{2p_y} \end{aligned}$$

Recall in lectures that the equivalent variation of a change in prices and income from (p^0, w^0) to (p^1, w^1) is defined as

$$\text{EV} = e\left(p^0, v\left(p^1, w^1\right)\right) - e\left(p^0, v\left(p^0, w^0\right)\right).$$

If $w^0 = w^1$ and the change in prices are caused by the imposition of commodity taxes then the deadweight loss (DWL) or excess burden of the taxes is given by

$$\text{DWL} = -\text{EV} - \sum_{l=1}^{L} t_l x_l\left(p^1, w^0\right),$$

where $t_l = p_l^1 - p_l^0$.

   (c) *Briefly explain why this measure may be viewed as a deadweight loss to (social) economic efficiency*
   $\sum_{l=1}^{L} t_l x_l(p^1, w^0)$ **is the tax revenue raised by the tax.** EV **is the equivalent change in wealth that would leave the individual as well off as the price change induced by the tax. So** $-\text{EV}$ **is how much tax revenue the government could have raised from the individual through a lump-sum (i.e. non-distortionary) tax, that would leave her as well off as under the commodity tax. So the difference (that is, the difference between how much the government could have raised through a lumpsum tax and how much it actually raised through distortionary taxes) measures the 'excess burden' associated with the distortionary commodity tax relative to the non-distortionary lumpsum tax on wealth.**

   (d) Suppose that the household initially faces prices $p^0 = (1,2)$ and has wealth $w^0 = 300$. Then a specific tax of 2 is imposed on bread (i.e. good x) that leads to its price rising to 3 (with the price of cheese, i.e. good y, and the household's wealth both remaining unchanged). *Calculate the DWL under the false assumption that the household demands come from just one consumer.*
   From (b) we have $x(p_x,p_y,w) = w/2p_x$ and $y(p_x,p_y,w) = w/2p_y$. Notice that this is the demand system corresponding to an individual with Cobb-Douglas (or equivalently, log-linear) utility, $u(x,y) = xy$ (or equivalently, $\ln x + \ln y$).

   $$\begin{aligned} v(p_x,p_y,w) &= x(p_x,p_y,w) \times y(p_x,p_y,w) \\ &= \frac{w^2}{4p_x p_y} \end{aligned}$$

From the identity

$$v\left(p_x, p_y, e\left(p_x, p_y, u\right)\right) \equiv u$$

we obtain

$$e\left(p_x, py, u\right) = 2\sqrt{p_x p_y u}$$

So

$$\begin{aligned} \text{EV} &= e\left(p^0, v\left(p^1, w^1\right)\right) - e\left(p^0, v\left(p^0, w^0\right)\right) \\ &= 2\sqrt{p_x^0 p_y^0 v\left(p_x^1, p_y^1, w\right)} - w^0 \\ &= 2\sqrt{\frac{2\left(300\right)^2}{4 \times 6}} - 300 \\ &= \left(\sqrt{3} - 3\right) \times 100 \end{aligned}$$

and

$$\sum_{l=1}^{L} t_l x_l\left(p^1, w^0\right) = 2 \times x\left(3, 2, 300\right) = 2 \times 50 = 100$$

So

$$\begin{aligned} \text{DWL} &= \left(3 - \sqrt{3}\right) \times 100 - 100 \\ &= \left(2 - \sqrt{3}\right) \times 100 \approx 26.795 \text{ (3 DP)} \end{aligned}$$

(e) *Using the individuals' indirect utility functions derived in part (a) calculate the two individual dead weight losses,* DWL_A *and* DWL_B. *Explain why* $\text{DWL}_A + \text{DWL}_B$ *does or does not equal* DWL.

Now from part (a) we have

$$\begin{aligned} v_A\left(p_x, p_y, w_A\right) &= x_A\left(p_x, p_y, w_A\right) = \frac{w_A}{p_x} \\ \text{so } e_A\left(p_x, p_y, u_A\right) &= p_x u_A \end{aligned}$$

Similarly,

$$\begin{aligned} v_B\left(p_x, p_y, w_B\right) &= x_B\left(p_x, p_y, w_B\right) = \frac{w_B}{p_y} \\ \text{so } e_B\left(p_x, p_y, u_B\right) &= p_y u_B. \end{aligned}$$

So

$$\begin{aligned} \text{EV}_A &= e_A\left(p^0, v_A\left(p^1, w_A^1\right)\right) - e\left(p^0, v_A\left(p^0, w_A^0\right)\right) \\ &= \frac{150}{3} - 150 = -100 \\ \text{EV}_B &= e_B\left(p^0, v_B\left(p^1, w_B^1\right)\right) - e\left(p^0, v_B\left(p^0, w_B^0\right)\right) \\ &= \frac{2 \times 150}{2} - 150 = 0 \end{aligned}$$

Since Andrew is the only individual who consumes bread, the amount of tax on bread that he pays must be 100, while Brenda is paying no tax at all since she consumes no bread. Thus

$$\begin{aligned} \text{DWL}_A &= 100 - 100 = 0 \\ \text{DWL}_B &= 0 - 0 = 0 \end{aligned}$$

So

$$\text{DWL}_A + \text{DWL}_B = 0 < \text{DWL}$$

Notice that since Andrew only consumes bread, a tax on bread is equivalent to a lumpsum tax for him, and so this generates no deadweight loss. And of course since Brenda does not consume bread, the tax on bread does not affect her consumption or her utility. But given the (arbitrary) equal division of wealth rule for this household, aggregate demand appears as if it is coming from a Cobb-Douglas utility function. For such preferences, a tax on bread is distortionary and so generates a deadweight loss, but this is only an artifact of the equal division of wealth rule and does not correspond to any underlying social planner allocating the household's wealth in order to maximize some social welfare function of the individuals' utilities. Thus we cannot attach any normative significance to the 'welfare' measures generated from this positive representative consumer. What appears to be a distortionary tax for the household is actually a non-distortionary tax, equivalent to a lumpsum for Andrew.