Rice University

Fall Semester Final Examination 2004

ECON501 Advanced Microeconomic Theory

ANSWER KEY

These answers are more detailed than I could possible expect from a student under examination conditions. I am making them available to aid your understanding of the material covered in ECON501 and help your preparation for the general exam at the end of this academic year.

PART A

1. [50 Points]

(a) Consider a preference relation that can be represented by the utility function

$$U(x_1, x_2, x_3, x_4) = x_1^{3/2} x_2^{3/2} + x_3^2 x_4.$$

i) Derive the uncompensated demand function for the utility maximization problem using this utility function.
Set up problem

$$\max_{x \geq 0} x_1^{3/2} x_2^{3/2} + x_3^2 x_4 \text{ s.t. } \sum_{i=1}^4 p_i x_i \leq w$$

Form Lagrangian

$$\mathcal{L}(x, \lambda, \mu) = x_1^{3/2} x_2^{3/2} + x_3^2 x_4 - \sum_{i=1}^4 \mu_i x_i - \lambda \left(\sum_{i=1}^4 p_i x_i - w \right)$$

Notice that $U(x_1, x_2, x_3, x_4)$ is sum of two convex functions $U^A(x_1, x_2) = x_1^{3/2} x_2^{3/2}$ and $U^B(x_3, x_4) = x_3^2 x_4$, hence optimal consumption will either have $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$, depending on which consumption bundle generates higher utility. So we can break the problem into two maximization of Cobb-Douglas utility problems

$$\begin{aligned} A &: \max_{(x_1, x_2) \geq 0} U^A(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq w \\ B &: \max_{(x_3, x_4) \geq 0} U^B(x_3, x_4) \text{ s.t. } p_3 x_3 + p_4 x_4 \leq w \end{aligned}$$

FONC for pblm A :

$$\begin{aligned} x_1 &: \frac{3U^A(x_1, x_2)}{2x_1} - \lambda p_1 \leq 0, \text{ with equality if } x_1 > 0 \\ x_2 &: \frac{3U^A(x_1, x_2)}{2x_2} - \lambda p_2 \leq 0, \text{ with equality if } x_2 > 0 \end{aligned}$$

We can see that optimal consumption for pblm A *entails* $x_1 > 0$ *and* $x_2 > 0$, *hence from two FONC we derive*

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow p_1 x_1 = p_2 x_2$$

Hence

$$x_1^A(p_1, p_2) = \frac{w}{2p_1} \text{ and } x_2^A(p_1, p_2) = \frac{w}{2p_2}$$

and so, the indirect utility for pblm A *is*

$$V^A(p_1, p_2, w) = \frac{w^3}{8p_1^{3/2} p_2^{3/2}}$$

By similar reasoning we have for FONC conditions for pblm B :

$$\begin{aligned} x_3 &: \frac{2U^A(x_1, x_2)}{x_3} - \lambda p_3 \leq 0, \text{ with equality if } x_3 > 0 \\ x_4 &: \frac{U^A(x_1, x_2)}{x_4} - \lambda p_4 \leq 0, \text{ with equality if } x_4 > 0 \end{aligned}$$

yielding

$$\frac{x_3}{x_4} = \frac{2p_4}{p_3} \Rightarrow p_3 x_4 = 2p_4 x_4$$

and so

$$x_3^B(p_3, p_4) = \frac{2w}{3p_3},\ x_4^B(p_3, p_4) = \frac{w}{3p_4} \text{ and } V^B(p_3, p_4, w) = \frac{4w^3}{27p_3^2 p_4}$$

So the indirect utility for the original problem is given by

$$\begin{aligned} V(p_1, p_2, p_3, p_4, w) &= \max\left(V^A(p_1, p_2, w), V^B(p_3, p_4, w)\right) \\ &= 4w^3 \max\left(\frac{1}{32(p_1 p_2)^{3/2}}, \frac{1}{27p_3^2 p_4}\right) \end{aligned}$$

Hence the uncompensated demands are given by

$$\begin{aligned} x_1(p_1, p_2, p_3, p_4, w) &= \begin{cases} 0 & \text{if } 32(p_1p_2)^{3/2} > 27p_3^2 p_4 \\ w/(2p_1) & \text{if } 32(p_1p_2)^{3/2} < 27p_3^2 p_4 \end{cases} \\ x_2(p_1, p_2, p_3, p_4, w) &= \begin{cases} 0 & \text{if } 32(p_1p_2)^{3/2} > 27p_3^2 p_4 \\ w/(2p_2) & \text{if } 32(p_1p_2)^{3/2} < 27p_3^2 p_4 \end{cases} \\ x_3(p_1, p_2, p_3, p_4, w) &= \begin{cases} 2w/(3p_3) & \text{if } 32(p_1p_2)^{3/2} > 27p_3^2 p_4 \\ 0 & \text{if } 32(p_1p_2)^{3/2} < 27p_3^2 p_4 \end{cases} \\ x_3(p_1, p_2, p_3, p_4, w) &= \begin{cases} w/(3p_4) & \text{if } 32(p_1p_2)^{3/2} > 27p_3^2 p_4 \\ 0 & \text{if } 32(p_1p_2)^{3/2} < 27p_3^2 p_4 \end{cases} \end{aligned}$$

and for the case in which $32(p_1p_2)^{3/2} = 27p_3^2 p_4$ *we have*

$$x(p_1, p_2, p_3, p_4, w) = \left\{\left(\frac{w}{2p_1}, \frac{w}{2p_2}, 0, 0\right), \left(0, 0, \frac{2w}{3p_3}, \frac{w}{3p_4}\right)\right\}$$

ii) Verify the indirect utility function from this problem satisfies all the requisite properties for an indirect utility function.
Notice that V^A (respectively, V^B) is decreasing in p_1 and p_2 (respectively, p_3 and p_4) and increasing in w. Furthermore, V^A (respectively, V^B) is homogeneous of degree zero in p_1 and p_2 (respectively, p_3 and p_4) and w.

$$\begin{aligned} V^A(\alpha p_1, \alpha p_2, \alpha w) &= \frac{(\alpha w)^3}{8(\alpha p_1)^{3/2}(\alpha p_2)^{3/2}} = \frac{w^3}{8p_1^{3/2}p_2^{3/2}} = V^A(p_1, p_2, w) \\ V^B(\alpha p_3, \alpha p_4, \alpha w) &= \frac{4(\alpha w)^3}{9(\alpha p_3)^2 \alpha p_4} = \frac{4w^3}{27p_3^2 p_4} = V^B(p_3, p_4, w) \end{aligned}$$

Hence

$$V(p_1, p_2, p_3, p_4, w) = \max\left(V^A(p_1, p_2, w), V^B(p_3, p_4, w)\right)$$

is non-decreasing in prices and increasing in w and homogeneous of degree zero in prices and wealth

$$\begin{aligned} V(\alpha p_1, \alpha p_2, \alpha p_3, \alpha p_4, \alpha w) &= \max\left(V^A(\alpha p_1, \alpha p_2, \alpha w), V^B(\alpha p_3, \alpha p_4, \alpha w)\right) \\ &= \max\left(V^A(p_1, p_2, w), V^B(p_3, p_4, w)\right) \\ &= V(p_1, p_2, p_3, p_4, w). \end{aligned}$$

To see that it is quasi-convex, by exploiting the homogeneity of degree 0 property, it is enough to show that

$$V^A(p_1, p_2, 1) = \frac{1}{8p_1^{3/2}p_2^{3/2}} \text{ and } V^B(p_3, p_4, 1) = \frac{4}{27p_3^2 p_4}$$

are both convex functions of prices, since this will mean V is a convex function of (normalized) prices as

$$V(p_1, p_2, p_3, p_4, 1) = \max\left(V^A(p_1, p_2, 1), V^B(p_3, p_4, 1)\right)$$

and the maximum of two convex functions is itself convex. So to see that $V^A(p_1, p_2, 1)$ and $V^B(p_3, p_4, 1)$ are convex, notice that their Hessians (matrices of second-order partial derivatives).

$$H^A = \begin{bmatrix} 15/(32p_1^3 p_2) & 9/(32p_1^2 p_2^2) \\ 9/(32p_1^2 p_2^2) & 15/(32p_1 p_2^3) \end{bmatrix} \text{ and } H^B = \begin{bmatrix} 24/(27p_3^4 p_4) & 8/(27p_3^3 p_4^2) \\ 8/(27p_3^3 p_4^2) & 8/(27p_3^2 p_4^3) \end{bmatrix}$$

are both positive definite matrices:

$$\begin{aligned} 15/\left(32p_1^3 p_2\right) &> 0, 15/\left(32p_1 p_2^3\right) > 0 \text{ and } \frac{225}{(32)^2 p_1^4 p_2^4} - \frac{81}{(32)^2 p_1^4 p_2^4} > 0 \\ 24/\left(27p_3^4 p_4\right) &> 0, 8/\left(27p_3 p_4^3\right) > 0 \text{ and } \frac{24 \times 8}{(27)^2 p_1^6 p_2^4} - \frac{8 \times 8}{(27)^2 p_1^6 p_2^4} > 0. \end{aligned}$$

iii) Does Roy's identity hold for this indirect utility function?
Explain or illustrate your answer.
Roy's identity holds wherever x is single-valued, (i.e. for all price-wealth combinations except for $8p_1p_2 = 9p_3p_4$)
I.e.

$$\begin{aligned}\frac{\partial V}{\partial w} &= \begin{cases} \partial V^B/\partial w & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ \partial V^A/\partial w & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases} \\ &= \begin{cases} 12w^2/\left(27p_3^2p_4\right) & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ 3w^2/\left[8\left(p_1p_2\right)^{3/2}\right] & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial V}{\partial p_1} &= \begin{cases} 0 & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ -3w^3/\left[16p_1^{5/2}p_2^{3/2}\right] & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases} \\ \frac{\partial V}{\partial p_2} &= \begin{cases} 0 & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ -3w^3/\left[16p_1^{3/2}p_2^{5/2}\right] & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial V}{\partial p_3} &= \begin{cases} -8w^3/\left(27p_3^3p_4\right) & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ 0 & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases} \\ \frac{\partial V}{\partial p_4} &= \begin{cases} -4w^2/\left(27p_3^2p_4^2\right) & \text{if } 32\left(p_1p_2\right)^{3/2} > 27p_3^2p_4 \\ 0 & \text{if } 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4 \end{cases}\end{aligned}$$

And by checking with the answer derived in part i) we see that

$$x_i = \frac{-\partial V/\partial p_i}{\partial V/\partial w}$$

as required for Roy's identity.

iv) Without working out the full utility maximization problem, show that a preference relation represented by the utility function

$$\hat{U}\left(x_1, x_2, x_3, x_4\right) = \max\left(\frac{3}{2}\ln x_1 + \frac{3}{2}\ln x_2, 2\ln x_3 + \ln x_4\right)$$

would also lead to the same uncompensated demand function.
Notice that

$$\begin{aligned}\hat{U}^A\left(x_1, x_2\right) &\equiv \frac{3}{2}\ln x_1 + \frac{3}{2}\ln x_2 \\ &= \ln U^A\left(x_1, x_2\right) \\ \hat{U}^B\left(x_1, x_2\right) &\equiv 2\ln x_3 + \ln x_4\end{aligned}$$

That is, both $\hat{U}^A$ and $\hat{U}^B$ are natural log transforms of U^A and U^B. Hence

$$\hat{U}^A > \hat{U}^B \Leftrightarrow U^A > U^B \Leftrightarrow 32\left(p_1p_2\right)^{3/2} < 27p_3^2p_4$$

Furthermore, the same consumption bundle (x_1^*, x_2^*) that is the solution of the UMP with $\hat{U}^A$ is the solution of the UMP with U^A, and the same consumption bundle (x_3^*, x_4^*) that is the solution of the UMP with $\hat{U}^B$ is the solution of the UMP with U^B. So the conditions under which the individual would choose a bundle like $(x_1, x_2, 0, 0)$ or a bundle like $(0, 0, x_3, x_4)$ are exactly the same for $\hat{U}$ as it was for U. Furthermore, when we choose positive amounts of goods 1 and 2 (respectively, goods 3 and 4) we choose the same amounts as we would under $\hat{U}$ as we did under U.

v) Briefly explain whether or not this means that U and $\hat{U}$ represent the same preferences.

No they are not. But where they differ would never be selected as solution to a UMP with linear prices.

(b) Suppose, there are two periods, 'today' (i.e. period 1) and 'tomorrow' (i.e. period 2), a single consumption good, and an individual called Zeek has preferences over two-period consumption streams that are *additively separable.* In particular assume his preferences over two-period consumption streams admit a representation of the form:

$$U(x_1, x_2) = u(x_1) + u(x_2)$$

Further suppose that Zeek is also a strictly *risk-averse* expected utility maximizer. Owing to a miscalculation, Art has caused Zeek damage for which Art is legally liable. Absent the damage, Zeek's income would have been the same in both periods, say z. The damage has reduced his income in period 1 by 10%. Zeek argues that in order to make him "whole" (as well off as he would have been had Art not damaged him), Art will have to pay Zeek more in the upcoming period than the amount Art caused Zeek to lose in the previous time period, because Zeek is risk averse.

No probabilities are mentioned above, but explain why risk aversion is relevant here. Is Zeek correct in his claim that since he is risk averse, Art's period 2 payment to him must be bigger than this period 1 loss to make him whole?

The fact that Zeek is an expected utility maximizer means there exists a von Neumann utility index $V : X \times X \to \mathbb{R}$, such that for any pair of lotteries over L and L' defined over $X \times X$

$$L \succsim L' \Leftrightarrow \sum_{(x_1,x_2)\in X\times X} L(x_1, x_2) V(x_1, x_2) \geq \sum_{(x_1,x_2)\in X\times X} L'(x_1, x_2) V(x_1, x_2)$$

Furthermore, we know that for any pair of degenerate lotteries $\delta_{(x_1,x_2)}$ and $\delta_{(x_1',x_2')}$ (i.e. certain two-period consumption streams)

$$\delta_{(x_1,x_2)} \succsim \delta_{(x_1',x_2')} \Leftrightarrow V(x_1, x_2) \geq V(x_1', x_2') \Leftrightarrow u(x_1) + u(x_2) \geq u(x_1') + u(x_2')$$

Hence $V(x_1, x_2)$ must be an increasing monotonic transformation of $U(x_1, x_2) = u(x_1) + u(x_2)$.

Risk aversion means Zeek prefers the mean of a lottery to the lottery itself. I.e. for any lottery L over $X \times X$ we have

$$\sum_{(x_1,x_2)\in X\times X} L(x_1, x_2) V(x_1, x_2) \geq V(\bar{x}_1, \bar{x}_2)$$

where

$$(\bar{x}_1, \bar{x}_2) = \sum_{(x_1,x_2)\in X\times X} L(x_1,x_2)(x_1,x_2)$$

is the expected consumption bundle. But by Jensen's inequality we know this is equivalent to $V(.,.)$ *being concave. But* $V(.,.)$ *being concave means that the preferences over certain two-period consumption streams must be convex – better than sets are convex sets. And so since* $U(x_1,x_2) = u(x_1) + u(x_2)$ *represents convex preferences, it follows that* $u(.)$ *is a concave function.*

Now concavity of $u(.)$ *implies*

$$\frac{1}{2}u(0.9\times z)+\frac{1}{2}u(1.1\times z) < u\left(\frac{0.9}{2}\times z+\frac{1.1}{2}\times z\right) = u(z) \text{ (by Jensen's inequality)}$$

Hence Zeek is right since the inequality in the line above implies

$$u(0.9\times z)+u(1.1\times z) < 2u(z) = u(z)+u(z).$$

(c) T-bone Pickens feeds his chickens on a mixture of soybeans and corn, depending on the prices of each. Assume the technology for 'producing' chickens is constant returns to scale. According to the data submitted by his managers, when the price of soybeans was \$10 a bushel and the price of corn was \$10 a bushel, they used 50 bushels of corn and 150 bushels of soybeans for each coop of chickens. When the price of soybeans was \$20 a bushel and the price of corn was \$10 a bushel, they used 300 bushels of corn and no soybeans per coop of chickens. When the price of corn was \$20 a bushel and the price of soybeans was \$10 a bushel, they used 250 bushels of soybeans and no corn for each coop of chickens.

i) Is there any evidence above that indicates Pickens's managers have not been minimizing costs?

The three points in the date correspond to points on three different isocost lines for a unit production. These three isocost curves correspond to the three straight lines given by

$$\begin{aligned} x_2 &= 200 - x_1 \\ x_2 &= 300 - 2x_1 \\ x_2 &= 125 - x_1/2 \end{aligned}$$

By graphing these three isocost lines we see that:

1. *input bundle* $(150, 50)$ *costs less than* $(0, 300)$ *and* $(250, 0)$ *when prices were* $(\$10, \$10)$.
2. *input bundle* $(0, 300)$ *costs less than* $(150, 50)$ *and* $(250, 0)$ *when prices were* $(\$20, \$10)$.
3. *input bundle* $(250, 0)$ *costs less than* $(150, 50)$ *and* $(0, 300)$ *when prices were* $(\$10, \$20)$.

Hence there is no evidence that Pickens's managers have not been minimizing costs.

ii) If Pickens's managers were always minimizing costs, briefly explain whether it is or is not possible to produce a coop of chickens using 50 bushels of soybeans and 150 bushels of corn?

At prices $(\$20, \$10)$ *the input bundle* $(50, 150)$ *costs* $\$2500 < \$3000$ *the cost of the input bundle* $(0, 300)$ *that was actually chosen when prices were* $(\$20, \$10)$. *Hence this input bundle must have been an infeasible bundle to reach the output target of* 1 *for if it were then the firm would not have been cost-mininimizing by choosing input bundle* $(0, 300)$ *when it did.*

PART B

2. [25 Points] Suppose there are 2 states of the world $s = 1, 2$ and a single consumption good. Assume the decision maker is a subjective expected utility maximizer with a Bernoulli utility index given by

$$u(c) = \frac{c^{1-\rho}}{1-\rho}$$

(a) What does it mean for this individual to be risk averse and what restriction do we have to place on ρ in order for this to be the case?

Risk aversion means the individual (weakly) prefers the mean of a lottery to the lottery itself. That is, for any state contingent consumption, if π *represents his belief that state one will obtain, then*

$$(\pi c_1 + [1-\pi] c_2, \pi c_1 + [1-\pi] c_2) \succsim (c_1, c_2) \text{ for all } (c_1, c_2) \in \mathbb{R}^2_+$$

Hence we require

$$\frac{(\pi c_1 + [1-\pi] c_2)^{1-\rho}}{1-\rho} \geq \frac{\pi c_1^{1-\rho} + [1-\pi] c_1^{1-\rho}}{1-\rho} \text{ for all } (c_1, c_2) \in \mathbb{R}^2_+$$

But notice, if $\rho < 0$, *then* $u(.)$ *is strictly convex since*

$$u''(c) = -\rho c^{-(1+\rho)} > 0$$

and so by Jensen's inequality we would have

$$u(\pi c_1 + [1-\pi] c_2) < \pi u(c_1) + (1-\pi) u(c_2)$$

So we can rule out $\rho < 0$. *On the other hand if* $\rho \geq 0$, *then* $u(.)$ *is concave and now Jensen's inequality gives us*

$$u(\pi c_1 + [1-\pi] c_2) \geq \pi u(c_1) + (1-\pi) u(c_2)$$

as required.

(b) Denoting state-contingent consumption bundles by (c_1, c_2), where c_1 represents the consumption in state 1 and c_2 represents the consumption in state 2, what implication can you draw if the bundle $(85, 45)$ is strictly preferred to $(45, 85)$? Devise a procedure that reveals the individual's subjective beliefs.

$(85, 45) \succ (45, 85)$ *means*

$$\pi u(85) + [1 - \pi] u(45) > \pi u(45) + [1 - \pi] u(85)$$

$$\Rightarrow (2\pi - 1) > u(85) - u(45) > 0 \Rightarrow \pi > 1/2$$

I.e. the individual believes state 1 is more likely than state 2.

Fix two outcomes, say 85 *and* 45. *Find a lottery* L, *of the form*

$$L(x) = \begin{cases} p & \text{if } x = 85 \\ 1 - p & \text{if } x = 45 \\ 0 & \text{otherwise} \end{cases}$$

such that

$$L \sim (85, 45)$$

Since he is a subjective expected utility maximizer, this indifference means that

$$pu(85) + (1 - p) u(45) = \pi u(85) + [1 - \pi] u(45)$$

$$\Rightarrow \pi = p$$

So set $\pi = p$ *(i.e.* p *is the individual's subjective belief about the likelihood of state 1 obtaining).*

(c) Suppose your procedure reveals that the individual's subjective belief is that state 1 is three times more likely than state 2. Illustrate her indifference map in state-contingent consumption space. Show her preference relation over state-contingent consumption bundles is *homothetic.*

Homotheticity requires,

$$(c_1, c_2) \sim (c_1', c_2') \Rightarrow (\lambda c_1, \lambda c_2) \sim (\lambda c_1', \lambda c_2') \text{ for all } \lambda > 0.$$

But

$$\begin{aligned}
& (c_1, c_2) \sim (c_1', c_2') \\
\Rightarrow\ & \frac{3}{4}\frac{c_1^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{c_2^{1-\rho}}{1-\rho} = \frac{3}{4}\frac{(c_1')^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{(c_2')^{1-\rho}}{1-\rho} \\
\Rightarrow\ & \lambda^{1-\rho}\left[\frac{3}{4}\frac{c_1^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{c_2^{1-\rho}}{1-\rho}\right] = \lambda^{1-\rho}\left[\frac{3}{4}\frac{(c_1')^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{(c_2')^{1-\rho}}{1-\rho}\right] \\
\Rightarrow\ & \frac{3}{4}\frac{(\lambda c_1)^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{(\lambda c_2)^{1-\rho}}{1-\rho} = \frac{3}{4}\frac{(\lambda c_1')^{1-\rho}}{1-\rho} + \frac{1}{4}\frac{(\lambda c_2')^{1-\rho}}{1-\rho} \\
\Rightarrow\ & (\lambda c_1, \lambda c_2) \sim (\lambda c_1', \lambda c_2') \text{ as required.}
\end{aligned}$$

Now suppose the problem the individual faces is to choose a portfolio of assets subject to her budget constraint. There are only two types of assets that she can choose. The first asset whose price is 1, is a 'risk-free' asset that yields a payoff of one unit of the consumption good if either state 1 or state 2 obtains. The second asset whose price is p, only pays one unit of the consumption good if state 1 obtains. If state 2 obtains, this second asset pays out zero.

(d) Formally set up the individual's portfolio problem letting a_1 denote her demand for asset 1 and a_2 her demand for asset 2.

Problem is

$$\max_{\langle a_1,a_2\rangle} \frac{3}{4}\frac{(a_1+a_2)^{1-\rho}}{1-\rho}+\frac{1}{4}\frac{(a_1)^{1-\rho}}{1-\rho} \text{ s.t. } a_1+pa_2=w$$

(e) For what range of p will $a_2(p,w)=0$ and for what range of p will $a_2(p,w)>0$? Will $a_1(p,w)$ ever be zero? Explain your answers.

FONCs

$$\begin{aligned}
a_1 &: \frac{3}{4}(a_1+a_2)^{-\rho}+\frac{1}{4}(a_1)^{-\rho}=\lambda \\
a_2 &: \frac{3}{4}(a_1+a_2)^{-\rho}=\lambda p \\
& a_1+pa_2=w
\end{aligned}$$

for $a_2=0$ *to be optimal we require*

$$\begin{aligned}
a_1 &: (a_1)^{-\rho}=\lambda \\
a_2 &: \frac{3}{4}(a_1)^{-\rho}=\lambda p \\
& a_1=w
\end{aligned}$$

I.e. $p=3/4$. *That is, the second asset has an actuarially fair price, so the individual invests solely in the safe asset and consumes the non-state consumption* w.

For $a_2>0$, *requires* $p<3/4$. *The asset represents a 'better-than-fair' bet, i.e. the expected payoff of investing 1 dollar in the asset is*

$$\frac{3/4}{p}>1$$

so the individual is willing to expose themselves to risk by purchasing positive amounts of the risky asset.

We can see from the FONC for a_1, *that* a_1 *would never be zero. Notice the only way to secure consumption in state 2 is through purchases of the safe asset. Thus if* $a_1=0$, *then* $c_2=0$, *but that cannot be utility maximizing as* $u'(c_2)\to\infty$ *as* $c_2\to 0$.

(f) Argue that neither asset is 'inferior' for the decision maker. That is, it is never the case that at given prices if wealth goes up the demand for either of the assets goes down.

You can show this on a diagram that if $(a_1 + a_2, a_1)$ *is the optimal consumption with wealth* w, *then the optimal state-contingent consumption with wealth* λw *is* $(\lambda [a_1 + a_2], \lambda a_1)$, *so* $a_i(1,p,w) = wa_i(1,p,1)$, $i = 1, 2$.

3. [25 Points] Consider an industry where a single output is produced using a single input. There are many technologies. Each technology can produce up to a common capacity of 1, which is *infinitesimally* small relative to the size of the market of this industry. Technologies are distinguished by a parameter b, where b is the (constant) *marginal product* of the input in this technology (up to capacity). That is, letting z, denote the quantity of input employed, the production function for technology b is:

$$F_b(z) = \begin{cases} b \times z & \text{if } z \leq 1/b \\ 1 & \text{if } z > 1/b \end{cases}$$

(a) Given the input price w and the output price p, solve the profit maximization problem of a firm with technology parameter b and derive its profit function $\pi_b(w,p)$.

For firm b, *problem is*

$$\max_{z \geq 0} py - wz \text{ s.t. } y \leq \min(bz, 1)$$

Notice if $pb > w$ *(the value of the marginal product is greater than the wage, then it is profitable to produce up to capacity). If* $pb < w$, *then the value of the marginal product doesn't cover the marginal cost (i.e. wage cost of labor), so it is profit maximizing for this technology to produce nothing. If* $pb = w$, *then the firm earns zero profit on its production, so it is indifferent between producing nothing, up to capacity or any amount in between. So solution is:*

$$z_b(w,p) = \begin{cases} 0 & \text{if } b < w/p \\ [0, 1/b] & \text{if } b = w/p \\ 1/b & \text{if } b > w/p \end{cases} \text{ and } y_b(w,p) = \begin{cases} 0 & \text{if } b < w/p \\ [0,1] & \text{if } b = w/p \\ 1 & \text{if } b > w/p \end{cases}$$

Hence the profit function for this firm is given by

$$\pi_b(w,p) = \begin{cases} 0 & \text{if } b \leq w/p \\ p - w/b & \text{if } b > w/p \end{cases}$$

Now assume technologies are distributed along $(0, \infty)$, according to the density function $h(b) = b^{-2}$. Hence the total capacity of the technologies that have a parameter b lying in the interval $[c, d]$ can be calculated by the expression

$$\int_c^d \frac{1}{b^2} db = \left[-\frac{1}{b} \right]_{b=c}^{b=d} = \frac{1}{c} - \frac{1}{d}$$

(b) Show the total industry profit is given by $\Pi(w,p) = p^2/(2w)$.

For given wage w and output price p, from (a) we saw for any firm with technology parameter $b > w/p$ makes positive profits and any firm with $b \leq 0$ makes zero profits, hence total industry profits are given by

$$\begin{aligned}
\Pi(w,p) &= \textstyle\int_0^\infty \pi_b(w,p)\, b^{-2} db \\
&= \textstyle\int_{w/p}^\infty \dfrac{(p - w/b)}{b^2} db \\
&= \left[-\dfrac{p}{b} + \dfrac{w}{2b^2}\right]_{b=w/p}^{b=\infty} \\
&= \dfrac{p}{w/p} - \dfrac{w}{2w^2/p^2} = \dfrac{p^2}{2w}
\end{aligned}$$

(c) Derive the industry's supply function.
Slow (laborious) way:

$$\begin{aligned}
z(w,p) &= \textstyle\int_0^\infty z_b(w,p)\, b^{-2} db \\
&= \textstyle\int_{w/p}^\infty \dfrac{1}{b^3} db \\
&= \left[-\dfrac{1}{2b^2}\right]_{b=w/p}^{b=\infty} \\
&= \dfrac{p^2}{2w^2}
\end{aligned}$$

and

$$\begin{aligned}
y(w,p) &= \textstyle\int_0^\infty y_b(w,p)\, b^{-2} db \\
&= \textstyle\int_{w/p}^\infty \dfrac{1}{b^2} db \\
&= \left[-\dfrac{1}{b}\right]_{b=w/p}^{b=\infty} \\
&= \dfrac{p}{w}
\end{aligned}$$

Short (smart) way: use Shephard's lemma.

$$\begin{aligned}
z(w,p) &= -\frac{\partial \Pi(w,p)}{\partial w} \\
&= -\frac{\partial}{\partial w}\left(\frac{p^2}{2w}\right) = \frac{p^2}{2w^2} \\
y(w,p) &= -\frac{\partial \Pi(w,p)}{\partial p} \\
&= -\frac{\partial}{\partial p}\left(\frac{p^2}{2w}\right) = \frac{p}{w}
\end{aligned}$$

(d) Derive the aggregate production function of this industry.

From part (c) we have

$$z = \frac{p^2}{2w^2} = \frac{1}{2}y^2 \Rightarrow F(z) = \sqrt{2z}$$

4. [25 Points] Suppose there are 1000 households who each desire one but only one video recorder and that each household is prepared to pay up to but no more than \$500. Suppose further, that there are a large number of potential manufacturers of video recorders, each with cost function

$$c(q) = q^2 + 40000$$

(a) If firms behave competitively, what is the long run equilibrium number of firms, price and quantity traded in this market?

In a perfectly competitive LR equilibrium with identical firms, long-run profits must be zero, hence

$$p = AC = MC$$

Notice that AC=MC means that those firms who are operating in the industry are doing so at their mininum efficient scale. To find the minimum efficient scale requires us to find the quantity q *where* $c(q)/q = c'(q)$, *that is,*

$$q + \frac{40000}{q} = 2q \Rightarrow q^2 = 40000 \Rightarrow q = 200$$

And at $q = 200$, $c'(200) = 2 \times 200 = 400$, *so the equilibrium price must be* $p = 400$. *Finally, since all 1000 households are willing to pay* \$500 *and the equilibrium price is* \$400, *demand would be* 1000. *So the number of firms operating in this perfectly competitive equilibrium is* $J = 1000/200 = 5$.

(b) Recalling that there are a large number of identical firms who may enter this industry, determine a (sub-game perfect, pure-strategy) Nash equilibrium of the two stage game where in the first stage firms determine whether to enter the market or not (and incur the fixed cost \$40,000 if they do enter) and in the second stage those who have entered play a one-shot Cournot quantity-setting game. How does the aggregate welfare in this equilibrium compare with the perfectly competitive outcome computed in (a)?

Let's start at a second stage in which J *firms have entered the industry. Given a belief (conjecture/theory) that the other* $J-1$ *firms will produce (and sell)* Q_{-j} *video-recorders in total, firm* j*'s profit maximizing problem may be expressed as*

$$\max_{q_j \in [0, 1000 - Q_{-j}]} 500q_j - q_j^2$$

(recall, \$40,000 *is a fixed cost which it has already incurred by entering, hence it is sunk).*

FONC

$$q_j : [500 - 2q_j][1000 - Q_{-j} - q_j] = 0$$

I.e.

$$q_j\left(Q_{-j}\right) = \min\left(250, 1000 - Q_{-j}\right)$$

Focussing on symmetric equilibria $q_j = q_{j'} = q$ *for all* j, j' *(but notice they are others) we have*

$$\begin{aligned} J = 1,2,3,4 &: \quad q = 250 \\ J \geq 5 &: \quad q = 1000/J \end{aligned}$$

And so, for each entrant, their anticipated profit given the number of firms who have entered is

$$\begin{aligned} J = 1,2,3,4: & \quad 500 \times 250 - (250)^2 - 40000 = 22500 \\ J \geq 5: & \quad 500 \times (1000/J) - (1000/J)^2 - 40000 \end{aligned}$$

So with the expectation that the market is shared equally in the second stage between those who choose to enter in the first stage, the equilibrium number of entrants is given by

$$\begin{aligned} 500 \times (1000/J^*) - (1000/J^*)^2 - 40000 &\geq 0 \\ 500 \times (1000/\left[J^* + 1\right]) - (1000/\left[J^* + 1\right])^2 - 40000 &< 0 \end{aligned}$$

Notice that for $J^* = 10$ *we have*

$$500 \times 100 - 100 \times 100 - 40000 = 0$$

So in the symmetric Cournot equilibrium we would have 10 entrants each producing and selling 100 *video-recorders and earning zero profits. Since the consumers are paying \$500 they also get no surplus, so aggregate welfare in this equilibrium is zero!*

(c) Design a revenue-neutral tax and rebate scheme that would implement the perfectly competitive outcome of part (a) even though producing firms act as Cournot competitors. In this context, take 'revenue-neutral' to mean that the equilibrium tax receipts equal the equilibrium rebate payments. (Hint: consider a per unit tax on production and a rebate to each household.)

Consider a per-unit tax of \$100 on the production of video-recorders and a lumpsum rebate of \$100 to each household. Since the rebate is lumpsum and we are doing a partial equilibrium analysis, even though their income has gone up by \$100, households are still only willing to pay up to \$500 for a video-recorder (i.e. the demand curve has not shifted). But the firm's marginal cost curve has shifted up by \$100 because of the specific tax levied on the production of video-recorders.

So now in the second stage, firm j's problem becomes:

$$\max_{q_j \in [0, 1000 - Q_{-j}]} 500q_j - q_j^2 - 100q_j$$

FONC

$$q_j : \left[400 - 2q_j\right]\left[1000 - Q_{-j} - q_j\right] = 0$$

I.e.

$$q_j\left(Q_{-j}\right)=\min\left(200,1000-Q_{-j}\right)$$

And again focussing on symmetric equilibria $q_j=q_{j'}=q$ *for all* j,j' *we have*

$$\begin{aligned} J=1,2,3,4,5 &: \; q=200 \\ J\geq 6 &: \; q=1000/J \end{aligned}$$

$$\begin{aligned} J &= 1,2,3,4,5: 500\times 200-\underbrace{(200)^2}_{\text{variable cost}}-\underbrace{40000}_{\text{fixed cost}}-\underbrace{100\times 200}_{\text{production tax}}=0 \\ J &> 5: 500\times(1000/J)-\underbrace{(1000/J)^2}_{\text{variable cost}}-\underbrace{40000}_{\text{fixed cost}}-\underbrace{100\times 1000/J}_{\text{production tax}}<0 \end{aligned}$$

So it is an equilibrium for five firms to decide to enter in the first stage and then each produce 200 video-recorders in the second stage. But notice that this is by no means the only equilibrium. Any number of firms from 1 *to* 5 *entering in the first stage and then each producing 200 video-recorders in the second stage is a subgame perfect equilibrium outcome!*