Rice University

Answers to Part B of First Semester Examination 2003

PART B

1. [**30 Points**] Mr. Croam lives for exactly two periods, $t = 0, 1$. Let $c_t \in \mathbb{R}$ denote his consumption in period t. Mr Croam's ($t = 0$) preferences over two-period consumption streams are represented given by the function

$$U(c_0, c_1) := u(c_0) + \delta E u(c_1)$$

where δ is a discount factor, u is an increasing strictly concave utility function, and the E denotes his expectation (at $t = 0$) about events in $t = 1$.

Initially, suppose that there is no uncertainty. Let $w_1 > 0$ be Mr. Croam's income in period 0 and let $w_1 \geq 0$ denote his income in period 1. Mr. Croam can save or borrow. Let $s \in \mathbb{R}$ denote his saving (notice that s could be negative) and let ρ denote the gross return on saving (i.e., $\rho = 1 + r$ if r is the interest rate). Thus, his consumption in period 0 is $w_0 - s$ and his consumption in period 1 is $w_1 + \rho s$. Assume interior solutions throughout.

(a) Write down necessary and sufficient conditions for Mr. Croam's chosen saving s^* to be greater than 0. (10 points).

Croam's pblm is

$$\max_{\langle c_0, c_1 \rangle} u(c_0) + \delta u(c_1) \text{ s.t. } c_0 = w_0 - s \text{ and } c_1 = w_1 + \rho s$$

FONC for s^*

$$-u'(w_0 - s^*) + \delta \rho u'(w_1 + \rho s^*) = 0$$

Hence for $s^* > 0$ require

$$-u'(w_0) + \delta \rho u'(w_1) > 0 \Leftrightarrow \rho > \frac{u'(w_0)}{\delta \rho u'(w_1)}$$

In words:. the gross return on saving (that is, the rate at which Mr Croam can transfer consumption from today to consumption tomorrow) must be greater than the his marginal rate of substitution at the endowment (w_0, w_1) (that is, the rate at which the he is willing to transfer consumption from today to consumption tomorrow).

(b) Suppose that $w_1 = 0$ and that the conditions from part (a) hold. Find a condition on the Mr. Croam's coefficient of relative risk aversion that is necessary and sufficient for s^* to be (locally) increasing in ρ. (10 points).

Differentiating the FONC from part (a) with respect to ρ we obtain

$$\delta \left[u''(\rho s^*) \left(\rho s^* + \rho^2 \frac{\partial s^*}{\partial \rho} \right) + u'(\rho s^*) \right] = -u''(w_0 - s^*) \frac{\partial s^*}{\partial \rho}$$

or

$$\left[-u''\left(\rho s^*\right)\rho^2\delta - u''\left(w_0 - s^*\right)\right]\frac{\partial s^*}{\partial \rho} = \delta\left[u''\left(\rho s^*\right)\left(\rho s^*\right) + u'\left(\rho s^*\right)\right]$$

Since from the FONC we have $u'\left(w_0 - s^*\right) = \delta\rho u'\left(\rho s^*\right) > 0$, it follows from the above inequality that:

$$\left[-\frac{u''\left(\rho s^*\right)\rho^2\delta}{\delta\rho u'\left(\rho s^*\right)} - \frac{u''\left(w_0 - s^*\right)}{u'\left(w_0 - s^*\right)}\right]\frac{\partial s^*}{\partial \rho} = \delta\left[\frac{u''\left(\rho s^*\right)\left(\rho s^*\right)}{\delta\rho u'\left(\rho s^*\right)} + \frac{u'\left(\rho s^*\right)}{\delta\rho u'\left(\rho s^*\right)}\right]$$

$$= \frac{1}{\rho}\left[\frac{u''\left(\rho s^*\right)\left(\rho s^*\right)}{u'\left(\rho s^*\right)} + 1\right]$$

Since

$$\left[-\frac{u''\left(\rho s^*\right)\rho^2\delta}{\delta\rho u'\left(\rho s^*\right)} - \frac{u''\left(w_0 - s^*\right)}{u'\left(w_0 - s^*\right)}\right] > 0 \text{ (recall } u' > 0, u'' < 0)$$

a necessary and sufficient condition for $\partial s^*/\partial\rho > 0$ is that

$$-\frac{u''\left(\rho s^*\right)\left(\rho s^*\right)}{u'\left(\rho s^*\right)} < 1.$$

Now suppose that Mr. Croam faces uncertainty over his period 1 income. Specifically, suppose that his period 1 income is given by $w_1 + \tilde{x}$ where $w_1 \geq 0$ and $E\tilde{x} = 0$. Let s^{**} denote Mr. Croam's new optimal saving.

(c) Show that if $u''' > 0$, then $s^{**} > s^*$ [Hint: Suppose that $s^{**} = s^*$ and compare the first order conditions.] (10 points).

Under uncertainty FONC becomes

$$-u'\left(w_0 - s^{**}\right) + \delta\rho\, \mathrm{E}\left[u'\left(w_1 + \tilde{x} + \rho s^{**}\right)\right] = 0$$

By Jensen's inequality, the strict convexity of u' (i.e. $u''' > 0$) implies

$$-u'\left(w_0 - s^*\right) + \delta\rho\, \mathrm{E}\left[u'\left(w_1 + \tilde{x} + \rho s^*\right)\right]$$
$$> -u'\left(w_0 - s^*\right) + \delta\rho u'\left(w_1 + \mathrm{E}\left[\tilde{x}\right] + \rho s^*\right)$$
$$= -u'\left(w_0 - s^*\right) + \delta\rho u'\left(w_1 + \rho s^*\right) = 0$$

Hence $s^{**} > s^*$

2. [**30 Points**] Consider a two-firm Cournot (quantity-competition) model with constant returns to scale but in which the firms' costs differ. Let c_j denote the firm j's cost per unit of output produced and assume $c_1 < c_2$. Let aggregate demand be given by $Q = 1 - p$.

   (a) Derive the Nash equilibrium of this model. Under what conditions does it involve only one firm producing? Which will it be? (10 points).

   Given firm i believes firm j is choosing to produce q_j^e, firm i's profit maximizing problem becomes

   $$\max_{q_i}\left(1 - q_j^e - q_i\right)q_i - c_i q_i$$

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FONC

$$1 - 2q_i - q_j^e - c_i \leq 0 \text{ with equality if } q_i > 0$$

$$\Rightarrow q_i = \max\left(\frac{1 - q_j^e - c_i}{2}, 0\right)$$

In a Nash equilibrium $q_i = q_i^e$, $i = 1, 2$. *So in the case where both are producing we have*

$$1 - 2q_1^* - q_2^* - c_1 = 0$$
$$1 - q_1^* - 2q_2^* - c_2 = 0$$

with the solution

$$q_1^* = \frac{1 - 2c_1 + c_2}{3} \text{ and } q_2^* = \frac{1 - 2c_2 + c_1}{3},$$
$$q_1^* + q_2^* = \frac{2 - c_1 - c_2}{3} \text{ and } p^* = \frac{1 + c_1 + c_2}{3}$$

Notice that when both firms are producing, $q_1^* - q_2^* = (c_2 - c_1)/3 > 0$, *so if one firm is only producing it will be firm 1. For firm 2 to choose* $q_2 = 0$, *entails*

$$q_1^e \geq 1 - c_2$$

But when $q_2^* = 0$, $q_1^* = (1 - c_1)/2$ *and* $p^* = (1 + c_1)/2$ *(the monopoly outcome for firm 1), hence we require*

$$\frac{1 - c_1}{2} \geq 1 - c_2 \Rightarrow c_1 \leq 2c_2.$$

(b) Show that if more than one firm is making positive sales show that we cannot have productive efficiency. In this situation what is the correct measure of (Marshallian) welfare loss relative to a fully efficient (that is, perfectly competitive) outcome. (10 points).

With linear demand and constant marginal cost, the usual DWL is the triangle with length $(1 - c_1 - (q_1^* + q_2^*))$ *[the difference between the competitive quantity* $1 - c_1$ *and the Cournot-Nash duopoly quantity* $(q_1^* + q_2^*)$ *and height* $(p^* - c_1)$ *(the difference between the duopoly price and the competive price which is of course the constant marginal cost of produciton). But now there is also productive inefficiency since the units produced by firm 2 could have been more cheaply produced by firm 1. Thus the expression for the deadweight loss is*

$$DWL = \frac{(1 - c_1 - (q_1^* + q_2^*))(p^* - c_1)}{2} + (q_1^* - q_2^*)(c_2 - c_1)$$
$$= \frac{(1 - 2c_1 + c_2)^2}{18} + \frac{(c_2 - c_1)^2}{3}$$

(c) Calculate the rate at which the Marshallian welfare changes as c_1 (respectively, c_2) changes. Can it ever be the case that a reduction in one of the firm's marginal cost *reduces* the Marshallian welfare in this market? (10 points).

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Marshallian welfare is the area under the demand curve above c_1 from 0 to $q_1^* + q_2^*$ less the production inefficiency which is the rectangle $(q_1^* - q_2^*) \times (c_2 - c_1)$. I.e.

$$
\begin{aligned}
W &= \frac{(1 - c_1 + p^* - c_1)(q_1^* + q_2^*)}{2} - (q_1^* - q_2^*)(c_2 - c_1) \\
&= \frac{(4 - 5c_1 + c_2)(2 - c_1 - c_2)}{18} - \frac{(c_2 - c_1)^2}{3}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial W}{\partial c_2} &= \frac{2 - c_1 - c_2 - 4 + 5c_1 - c_2}{18} - \frac{2(c_2 - c_1)}{3} \\
&= \frac{-1 + 8c_1 - 7c_2}{9} = \frac{-(1 - c_1) - 7(c_2 - c_1)}{9} < 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial W}{\partial c_1} &= \frac{-5(2 - c_1 - c_2) - (4 - 5c_1 + c_2)}{18} + \frac{2(c_2 - c_1)}{3} \\
&= \frac{-14(1 - c_2) + 2(c_2 - c_1)}{18}.
\end{aligned}
$$

Hence if $(c_2 - c_1) > 7(1 - c_2)$, then a reduction in firm 2's marginal cost will reduce Marshallian welfare in this market.

3. [**30 Points**] A firm's production of bacon generates a smelly gas as an unpleasant side product. Let $c(y, m; \mathbf{w})$ denote the (minimum) input cost of producing y tons of bacon and m cubic meters of gas when input prices are given by the vector $\mathbf{w} \gg \mathbf{0}$. Let $p > 0$ denote the price of bacon. Assume that $\partial c / \partial y > 0$, $\partial c / \partial m < 0$ and that $c(.,.;\mathbf{w})$ is strictly convex in y and m. Let stars * denote solutions and assume throughout that $y^* > 0$.

   (a) Show that $c(y, m; .)$ is concave in $\mathbf{w}$. (10 points).
   Fix two input price vectors $\mathbf{w}$ and $\mathbf{w}'$ and consider $\mathbf{w}'' = \alpha \mathbf{w}' + (1 - \alpha)\mathbf{w}''$, for some α in $(0, 1)$. Let $\mathbf{x}$ (respectively, $\mathbf{x}'$ and $\mathbf{x}''$) be the min. input cost bundle for $\mathbf{w}$ (respectively, $\mathbf{w}'$ and $\mathbf{w}''$). By cost minimization we have

$$
\begin{aligned}
c(y, m; \mathbf{w}'') &= \alpha \mathbf{w}.\mathbf{x}'' + (1 - \alpha)\mathbf{w}'.\mathbf{x}'' \\
&\geq \alpha \mathbf{w}.\mathbf{x} + (1 - \alpha)\mathbf{w}'.\mathbf{x}' \\
&= \alpha c(y, m; \mathbf{w}) + (1 - \alpha) c(y, m; \mathbf{w}'')
\end{aligned}
$$

   So $c(y, m; \mathbf{w})$ is concave in $\mathbf{w}$.

   (b) Suppose that the government imposes a ceiling on gas emissions such that $m \leq \bar{m}$. Assuming that this constraint binds, write down the firm's profit maximization problem with respect to y, and find necessary and sufficient conditions for $\partial y^* / \partial \bar{m} > 0$. (10 points).
   Profit maximizing problem is

$$
\max_{\langle y, m \leq \bar{m} \rangle} py - c(y, m; \mathbf{w})
$$

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If constraint binds then FONC for output choice y is

$$y : p = \frac{\partial c\left(y^*, \bar{m}; \mathbf{w}\right)}{\partial y} \quad \text{(i.e. price equals marginal cost)}$$

Differentiating the FONC wrt $\bar{m}$

$$0 = \frac{\partial^2 c\left(y^*, \bar{m}; \mathbf{w}\right)}{\partial y^2} \frac{\partial y^*}{\partial \bar{m}} + \frac{\partial^2 c\left(y^*, \bar{m}; \mathbf{w}\right)}{\partial m \partial y}$$

So

$$\frac{\partial y^*}{\partial \bar{m}} = -\frac{\frac{\partial^2 c(y^*, \bar{m}; \mathbf{w})}{\partial m \partial y}}{\frac{\partial^2 c(y^*, \bar{m}; \mathbf{w})}{\partial y^2}}$$

Since c strictly convex in y the denominator is positive, so a necessary and sufficient condition for $\partial y^*/\partial \bar{m} > 0$ is

$$\frac{\partial^2 c\left(y^*, \bar{m}; \mathbf{w}\right)}{\partial m \partial y} < 0$$

*that is, an increase in m reduces the **marginal cost** of production.*

(c) Suppose now that the government abandons its emissions ceiling and replaces it with a tax $t > 0$ on gas emissions. Thus, the new cost of producing (y, m) is given by $c(y, m; \mathbf{w}) + tm$. Write down the firm's profit maximization problem with respect to y and m. Show that maximized profits are convex in t, and that $\partial m^*/\partial t \leq 0$. (10 points).

Suppose (y, m), (y', m') and $(y'', m''))$ maximize profits for t, t' and t'', respectively, where $t'' = \alpha t + (1 - \alpha) t'$, for some α in $(0, 1)$. By profit maximization it follows

$$
\begin{aligned}
\pi\left(p, \mathbf{w}, t\right) &= py - c\left(y, m; \mathbf{w}\right) - tm \geq py'' - c\left(y'', m''; \mathbf{w}\right) - tm'' \\
\pi\left(p, \mathbf{w}, t'\right) &= py' - c\left(y', m; \mathbf{w}'\right) - tm \geq py'' - c\left(y'', m''; \mathbf{w}\right) - t'm''
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\alpha\pi\left(p, \mathbf{w}, t\right) + (1 - \alpha)\pi\left(p, \mathbf{w}, t'\right) \\
\geq\ & py'' - c\left(y'', m''; \mathbf{w}\right) - \left[\alpha t + (1 - \alpha)t'\right]m'' \\
=\ & py'' - c\left(y'', m''; \mathbf{w}\right) - t''m'' = \pi\left(p, \mathbf{w}, t''\right)
\end{aligned}
$$

That is, $\pi\left(p, \mathbf{w}, t\right)$ is convex in t.

Let $\mathbf{x}$ and $\mathbf{x}'$ be the input vectors for the profit maximizing plans associated with t and t' respectively. By profit maximization

$$\pi\left(p, \mathbf{w}, t\right) = py - \mathbf{w}.\mathbf{x} - tm \geq py' - \mathbf{w}.\mathbf{x}' - tm'$$

$$\Rightarrow -p\left(y' - y\right) + \mathbf{w}.\left(\mathbf{x}' - \mathbf{x}\right) + t\left(m' - m\right) \geq 0$$

and

$$\pi\left(p, \mathbf{w}, t'\right) = py' - \mathbf{w}.\mathbf{x}' - t'm' \geq py - \mathbf{w}.\mathbf{x} - t'm$$

$$\Rightarrow p\left(y'-y\right)-\mathbf{w}.\left(\mathbf{x}'-\mathbf{x}\right)-t\left(m'-m\right)\geq 0$$

So by adding these two inequalities we get

$$-\left(t'-t\right)\left(m'-m\right)\geq 0$$

Or in differential terms

$$\partial m^{*}/\partial t\leq 0$$