ECON501 Advanced Microeconomic Theory 1
Fall Semester 2007
Problem Set 7

The due date for this problem set is Wednesday November 28.

1. Consider the production function $f(z) : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $f(z_1, z_2) = z_1^a z_2^b$.

   (a) Find the MRTS.

   (b) Find restrictions on the parameters a and b such that the production set Y exhibits constant returns to scale.

   (c) If $a = .5$ and $b = .6$,is the production set convex? Prove your answer.

   (d) Show that the isoquants for any $0 < a, b < 1$ are convex. Does this imply that the production function is concave for $0 < a, b < 1$? Prove your answer.

2. Given is the CES production function $f(z) : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $f(z_1, z_2) = (az_1^\rho + bz_2^\rho)^{\frac{1}{\rho}}$.

   (a) Find the MRTS. For which values of ρ, a and b are the isoquants convex?

   (b) Does this technology exhibit constant returns to scale?

3. **Homothetic Production Functions**. If $h(z)$ is homogenous of degree 1 then $f(z) = g(h(z))$ is homothetic if $g(.)$ is an increasing function of one variable. Homogeneous functions are a subset of the set of homothetic functions. (Note that for the purpose of production functions we need to make sure that $g(x)$ is a positive monotonic function for $x \geq 0$.) An **identifying characteristic** of a homothetic function is that its MRTS is the same at z and tz $\forall t > 0$.

   (a) Verify that the MRTS of $h(z)$ and of $f(z)$ are the same.

   (b) Given the production function

   $$f(z) = 3az_1^{.5}bz_2^{.5}\left(2 + az_1^{.5} + bz_2^{.5}\right) + az_1^{.5}\left(1 + 3az_1^{.5} + a^2 z_1\right) + bz_2^{.5}\left(1 + 3bz_2^{.5} + b^2 z_2\right).$$

   Show that this is a homothetic function.

4. Consider the production function in question 1 again.

   (a) Set up the profit maximizing problem and derive the Kuhn-Tucker conditions.

   (b) Show that if $a + b = 1$, the profit maximizing output is for some price vectors $(p, w_{1,} w_2)$ equal to $zero$, for others the optimal supply is anything from $zero$ to $infinity$, and yet again for others it is $infinity$.

   (c) Suppose $a, b > 0$ and $a + b < 1$. Find the profit function, the supply function and the factor demands. Verify that the profit function is convex in $(p, w_{1,} w_2)$, and that it is homogenous of degree 1. Verify that the factor demands are homogenous of degree zero.

5. Consider the production function given in 2) again. Assume $a, b > 0$, and $\rho \in (0, 1)$.

   (a) Set up the profit maximizing problem and derive the Kuhn-Tucker conditions.

   (b) Show that the second-order derivative matrix of the production function is negative semidefinite.

   (c) Show that the profit maximizing output is for some price vectors $(p, w_{1,} w_2)$ equal to $zero$, for others the optimal supply is anything from $zero$ to $infinity$, and yet again for others it is $infinity$

6. **Profit Maximization**

   Let $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ be the production function for a good using inputs $(z_1, \ldots, z_{L-1})$. Suppose the marginal product of each input is everywhere non-negative and that f is strictly concave. Prove the following.

   (a) An increase in the price of the output increases the profit maximising level of output.

   (b) An increase in the price of the output increases the demand for some input.

   (c) An increase in the price of an input that is currently being used at a postive level, leads to a reduction in the demand for that input.

7. **Variable and long run costs**

   A firm uses two inputs, capital and labour to produce a output. The firm is a price-taker in both the input and output markets. With its capital stock fixed at K, the variable costs for the firm are given by
   $$C_v(y, K) = way/K + wmy$$
   where y is the level of output, w is the wage-rate for labour, and a and m are positive constants. The factor price for capital is r per unit of capital employed. Thus total costs, given y and K, are
   $$T(y, K) = C_v(y, K) + rK$$

   (a) The long run cost curve is defined as the minimum if total costs with respect to K for a given y. Derive the long run cost curve $C(y)$ for this firm.

   (b) Show that the long run cost curve has declining average and marginal costs.

   (c) The variable cost function is derived from the production function and provides a "dual" description of the production technology. What is the production function that gives rise to the variable cost function above? (Hint: with only two factors of production, one of which, capital, is fixed in the short run, $C_v(y, K)$ is equal to the wage bill, that is, wL.)

8. **Aggregating "Many (Infinitesimally) Small" Firms**

   One output is produced from two inputs, call them labour and capital. There are many technologies. Each technology can produce up to a common capacity, say 1 unit, (but no more) with fixed and proportional input requirements a_L and a_K (that is, the capacity level of output requires a_L units of labour and a_K units of capital.) So a technology is characterised by $a = (a_L, a_K)$ in $\mathbb{R}^2_+$.

   Assume this common capacity of each technology is "small" relative to the market. Define the total capacity of a set of technologies as the *area of that set in* $\mathbb{R}^2_+$. For example, the total production that is feasible for technologies in the set $S = \left\{(a_L, a_K) \text{ in } \mathbb{R}^2_+ : a_L + a_K \leq 4\right\}$ can be calculated by the double integral

$$\begin{aligned}\int_0^4 \int_0^{4-a_K} 1 da_L da_K &= \int_0^4 \left[a_L\right]_{a_L=0}^{a_L=4-a_K} da_K \\ &= \int_0^4 (4-a_K)\, da_K \\ &= \left[4a_K - a_K^2/2\right]_{a_L=0}^{a_L=4} = 8.\end{aligned}$$

(a) Given the input prices (w, r) and output price $p = 1$, solve the profit maximisation problem of a firm with technology (a_L, a_K). (Hint: this is a very easy question, so if you are having difficulty try and forget all the economics you ever learnt in this course or in your previous studies and try to solve the problem from 'first-principles'.)

(b) Compute the total, that is, aggregate profits for $(w, r) = (1, 2)$. More generally, find the (aggregate) profit function $\Pi(1, w, r)$ for $(w, r) > (0, 0)$. (Hint: what is the set of profitable technologies? Add, or more precisely, integrate, all the profits earned by these profitable technologies to obtain the total or aggregrate profit.)

(c) For a given $(w, r) > (0, 0)$, integrate over the set of profitable technologies to obtain the input (derived) demand function for labour and capital . Check your answer is correct by using Hotelling's lemma to obtain the input demands from the profit function you derived in 8b)

(d) Derive the aggregate production function using the profit function obtained in 8b). (Hint: refer to property 4(b) of the profit function from your class notes for the method. The production function so derived should be $f(L, K) = AL^{1/3}K^{1/3}$, where A is a positive constant. That is, you should obtain that the aggregate production set can be characterised by a decreasing returns to scale Cobb-Douglas production function!)